IV. *On the Theory of Lubrication and its Application to Mr.* Beauchamp Tower's *Experiments, including an Experimental Determination of the Viscosity of Olive Oil.*

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[Plate 8.]

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Section I.—Introductory.

1. Lubrication, or the action of oils and other viscous fluids to diminish friction and wear between solid surfaces, does not appear to have hitherto formed a subject for theoretical treatment. Such treatment may have been prevented by the obscurity of the physical actions involved, which belong to a class as yet but little known, namely, the boundary or surface actions of fluids; but the absence of such treatment has also been owing to the want of any general laws discovered by experiment.

The subject is of such fundamental importance in practical mechanics, and the opportunities for observation are so frequent, that it may well be a matter of surprise that any general laws should have for so long escaped detection.

Besides the general experience obtained, the friction of lubricated surfaces has been the subject of much experimental investigation by able and careful experimenters. But, although in many cases empirical laws have been propounded, these fail for the most part to agree with each other and with the more general experience.

2. The most recent investigation is that of Mr. Beauchamp Tower, undertaken at the instance of the Institution of Mechanical Engineers. Mr. Tower's first report was published November, 1883, and his second report in 1884 ("Proc. Inst. Mechanical Engineers").

In these reports Mr. Tower, making no attempt to formulate, states the results of experiments apparently conducted with extreme care and under very various and well-chosen circumstances. Those results which were obtained under the ordinary conditions of lubrication so far agree with the results of previous investigators as to show a want of any regularity. But one of the causes of this want of regularity, irregularity in the supply of the lubricant, appears to have occurred to Mr. Tower early in his investigation, and led him to include amongst his experiments the unusual circumstances of surfaces completely immersed in oil. This was very fortunate, for not only do the results so obtained show a great degree of regularity, but while making these experiments he was accidentally led to observe a phenomenon which, taken with the results of his experiments, amounts to a crucial proof that in these experiments with the oil bath the surfaces were completely and continuously separated by a film of oil; this film being maintained by the motion of the journal, although the pressure in the oil at the crown of the bearing was shown by actual measurement to be as much as 625 lbs. per sq. inch above the pressure in the oil bath.

These results obtained with the oil bath are very important, notwithstanding that the condition is not common in practice. They show that with perfect lubrication a definite law of variation of the friction with the pressure and velocity holds for a particular journal and brass. This strongly implies that the irregularity previously found was due to imperfect lubrication. Mr. Tower has brought this out:—Substituting for the bath an oily pad, pressed against the free part of the journal, and making it so slightly greasy that it was barely perceptible to the touch, he again found considerable regularity in the results; these, however, were very different from those with the bath. Then with intermediate lubrication he obtained intermediate results, of which he says:—"Indeed, the results, generally speaking, were so uncertain and irregular that they may be summed up in a few words. The friction depends on the quantity and uniform distribution of the oil, and may be anything between the oil bath results and seizing, according to the perfection or imperfection of the lubrication."

3. On reading Mr. TOWER's report it occurred to the author as possible that in the case of the oil bath the film of oil might be sufficiently thick for the unknown boundary actions to disappear, in which case the results would be deducible from the equations of hydrodynamics. Mr. TOWER appears to have considered this, for he remarks that according to the theory of fluid friction the resistance would be as the square of the velocity, whereas in his results it does not increase according to this law. Considering how very general the law of resistance as the square of the velocity is with fluids, there is nothing remarkable in the assumption of its holding in such a case. But the study of the behaviour of fluid in very small channels, and particularly the recent determination by the author of the critical velocity at which this law changes from that of the square of the velocity to that of the simple ratio, shows that with such highly viscous fluids as oils, such small spaces as those existing between the journal and its bearing, and such limited velocity as that of the surface of the journal, the resistance would vary, *cæteris paribus*, as the velocity. Further, the thickness of the oil film would not be uniform and might be affected by the velocity, and as the resistance would vary, *cæteris paribus*, inversely as the thickness of the film, the velocity might exert in this way a secondary effect on the resistance; and, further still, the resistance would depend on the viscosity of the oil, and this depends on the temperature. But as Mr. TOWER had been careful to make all his experiments in the same series with the journal at a temperature of 90° Fahr., it did not at first appear that there could be any considerable temperature effect in his results.

4. The application of the hydrodynamical equations to circumstances similar in so far as they were known to those of Mr. TOWER's experiments, at once led to an equation between the variation of pressure over the surface and the velocity, which equation appeared to explain the existence of the film of oil at high pressure. This equation was mentioned in a paper read before Section A. of the British Association at Montreal, 1884. It also appears from a paragraph in the President's Address (Brit. Assoc. Rep., 1884, p. 14) that Professor STOKES and Lord RAYLEIGH had simultaneously arrived at a similar result. At that time the author had no idea of attempting its integration. On subsequent consideration, however, it appeared that the equation might be transformed so as to be approximately integrated, and the theoretical results thus definitely compared with the experimental.

5. The result of this comparison was to show that with a particular journal and

brass the mean thickness of the film of oil would be sensibly constant, and hence, if the viscosity was constant, the resistance would increase directly as the speed. As this was not in accordance with Mr. TOWER's experiments, in which the resistance increased at a much slower rate, it appeared that either the boundary actions became sensible or that there must have been a rise in the temperature of the oil which had escaped the thermometers used to measure the temperature of the journal.

That there would be some excess of temperature in the oil film on which all the work of overcoming the friction is spent is certain; and after carefully considering the means of escape of this heat, it seems probable that there would be a difference of several degrees between the oil bath and the film of oil.

This increase of temperature would be attended by a diminution of viscosity, so that as the resistance and temperature increased with the velocity the viscosity would diminish and cause a departure from the simple ratio.

6. In order to obtain a quantitative estimate of these secondary effects, it was necessary to know exactly the relation between the viscosity and temperature of the lubricant used. For this purpose an experimental determination was made of the viscosity of olive oil at different temperatures as compared with the known viscosity of water. From the results of these experiments an empirical formula has been deduced, by means of which definite expressions have been obtained for the approximate variation of the viscosity with the speed and load. Taking these variations of viscosity into account, the results obtained from the hydrodynamical theory are brought into complete accordance with these experiments of Mr. TOWER. Thus we have not only an explanation of the very novel phenomena brought to light by these experiments, and what appears to be an important verification of the assumptions on which the theory of hydrodynamics is founded, but we also find, what is not shown in the experiments, how the various circumstances under which the experiments have been made affect the results.

7. Two circumstances particularly are brought out in the theory as principal circumstances which seem to have hitherto entirely escaped notice, even that of Mr. TOWER.

One of these is the difference in the radii of the journal and of the brass or bearing.

It is well known that the fitting between the journal and its bearing produces a great effect on the carrying power of the journal, but this fitting is rather supposed to be a matter of smoothness of surface than a degree of correspondence in radii. The radius of the bearing must always be as much larger than that of the journal as is necessary to secure an easy fit; but more than this, I think, has never been suggested.

Now it appears from the theory that if viscosity were constant the friction would be inversely proportional to the difference in radii of the journal and the bearing, and this although the arc of contact is less than the semicircumference. Taking the temperature into account, it appears from the comparison of the theoretical results with the experimental that at a temperature of 70·5° Fahr. the radius of one of the brasses used was ·00077 inch greater than that of the journal, while at a temperature of 70° Fahr. that of the other was ·00084 inch, or 9 per cent. larger than the first.

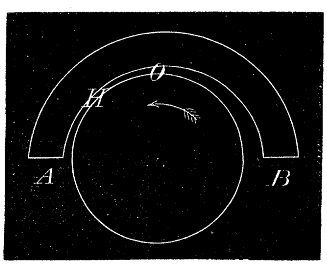
These two brasses were probably both bedded to the journal in the same way, and had neither of them been subjected to any great amount of wear, so that there is nothing surprising in their being so nearly the same fit. It would be extremely interesting to find whether prolonged wear of the brass tends to preserve or destroy the fit. This does not appear from Mr. TOWER's experiments. It does appear, however, that the brass expands with an increase of temperature more than the journal, and that its radius increases as the load increases in a very definite manner.

Another circumstance brought out by the theory, and remarked on both by Lord RAYLEIGH and the author at Montreal, but not before expected, is that the point of nearest approach of the journal to the brass is not by any means in the line of the load, and, what is still more contrary to common supposition, is on the *off** side of the line of load.

This circumstance, the reason for which is rendered perfectly clear by the conditions of equilibrium, at once accounts for a singular phenomena mentioned by Mr. TOWER, viz., that the journal having been run in one direction until the initial tendency to heat had entirely disappeared, on being reversed it immediately began to heat again; but this effect stopped when the process had been often repeated. The fact being that running in one direction the brass had been worn to the journal only on the *off* side for that direction, so that when the motion was reversed the new *off* side was like a new brass.

7A. The circumstances which determine the greatest load which a bearing will carry with complete lubrication, *i.e.*, with the film of oil extending between brass and journal throughout the entire arc, are definitely shown in the theory.

The effect of increasing the load beyond a certain small value being to cause the brass to approach nearer to the journal at a point H which moves from A towards O as the load increases, and when the load is such that the least separating distance ca is about half the difference of radii a, the angular position of H is 40° to the off side of O, the middle of the brass. At this point the pressure in the oil film is everywhere greater



than at A and B, the extremities of the brass, but when the load further increases the pressure towards A on the off side becomes smaller or negative. This when sufficient

* "On" and "off" sides of the line of load are used by Mr. TOWER to express respectively the sides of approach and succession, as B and A in the figure, the arrow indicating the direction of rotation.

will cause rupture in the oil film, which will then only bextend etween the brass and journal over a portion of the whole arc and a smaller portion as the load increases. Thus, since the amount of negative pressure which the oil will bear depends on circumstances which are uncertain, the limit of the safe load for complete lubrication is that which causes the least separating distance to be half the difference of radii of the brass and journal.

The rupture of the oil film does not take place at the point of nearest approach, and hence the brass may still be entirely separate from the journal, and could the integrations be effected it would be possible to deal as definitely with this condition as with that of complete lubrication; but these difficulties have limited the actual application of the theory to complete lubrication. This however by no means requires an oil bath, but merely sufficient oil on the journal.

What happens when the supply of oil is limited, *i.e.*, insufficient for complete lubrication, cannot be definitely expressed without further integrations; but sufficient may be seen to show that the brass will still be completely separated from the journal, although the separating film will not touch the brass, except over a limited area; but in this case it is easy to show by general reasoning that in the one extreme, where the supply of oil is limited, the friction increases directly as the load and is independent of the velocity, while in the other, where the oil is abundant, the circumstances are those of the oil bath.

The effect of the limited length of the journal is also apparent in the equations, as is also the effect of necking the shaft to form the journal, so that the ends of the brass are against flanges on the shaft.

The theory is perfectly applicable to cases in which the direction of the load on the bearing varies, as with the crank pin and with the bearings of the crank shaft of the steam-engine; but these cases have not been considered, as there are no definite experiments to compare.

8. Although in the main the present investigation has been directed to the circumstances of Mr. TOWER's experiments, viz., a cylindrical journal revolving in a cylindrical brass, it has, on the one hand, been found necessary to proceed from the general equations of equilibrium of viscous fluids, and, on the other hand, to consider somewhat generally the physical property of viscosity and its dependence on temperature.

The property of viscosity has been discussed at length in Section II.; which section also contains the account of an experimental investigation as to the viscosity of olive oil.

The general theory deduced from the hydrodynamical equations for viscous fluids, with the methods of application, is given in Sections IV., V., VI., VII., and VIII.

As there are some considerations which cannot be taken into account in the more general method, which method also tends to render obscure the more immediate purpose of the investigation, a preliminary discussion of the problem, illustrated by aid of the graphic method, has been introduced as Section III. Finally, the definite application of the theory to Mr. TOWER's experiments is given in Section IX.

SECTION II.—THE PROPERTIES OF LUBRICANTS.

9. *The Definition of Viscosity.*

In distinguishing between solid and fluid matter, it is customary to define fluid as a state of matter incapable of sustaining tangential or shearing stress. This definition, however, as is well known, is only true as applied to *actual* fluids when at rest. The resistance encountered by water and all known fluids flowing steadily along parallel channels, affords definite proof that in certain states of motion all actual fluids will sustain shearing stress. These actual fluids are, therefore, called in the language of mathematics imperfect or viscous fluids.

In order to obtain the equations of motion of such fluids, it has been necessary to define clearly the property of viscosity. This definition has been obtained from the consideration that to cause shearing stress in a body it is necessary to submit it to forces tending to change its shape. Forces tending to cause a general motion, whether linear, revolving, uniform expansion, or uniform contraction, call forth no shearing stress.

Using the term *distortion* to express change of shape, apart from change of position, uniform expansion, or contraction, the viscosity of a fluid is defined as the shearing stress caused in the fluid while undergoing distortion, and the shearing stress divided by the rate of distortion is called the coefficient of viscosity, or, commonly, the viscosity of the fluid.

This is best expressed by considering a mass of fluid bounded by two parallel planes at a distance a, and supposing the fluid between these planes to be in motion in a direction parallel to these surfaces with a velocity which varies uniformly from 0 at one of these surfaces to u at the other. Then the rate of distortion is

$$\frac{u}{a}$$

and the shearing stress on a plane parallel to the motion is expressed by

$$f=\mu\frac{u}{a} \quad \ldots \ldots \ldots \ldots \quad (1)$$

μ being the coefficient of viscosity or the modulus of the resistance to distortional motion.

10. *The Character of Viscosity.*

In dealing with ideal fluids, it is of course allowable to consider μ as being zero or having any conceivable value; but practically, as regards natural philosophy, the value of any such considerations depends on whether the calculated behaviour of the ideal fluid is found to agree with the behaviour of the actual fluids—whether taking a particular fluid, a value of μ can be found such that the values of f calculated by equation (1) agree with the values of f determined by experiments for all values of a and u.

In the mathematical theory of viscous fluid, μ is assumed to be constant for a particular fluid. This supposition is sometimes justified by reference to some assumed dynamical constitution of fluids; but apart from such hypotheses there is no more ground for supposing a constant value for μ than there is for supposing a particular law of gravitation, in other words, there is no ground at all. If a particular value of μ is found to bring the calculated results into agreement with all experimental results, then this value of μ defines a property of actual fluids, and of course it has been with this object that the mathematical theory of μ has been studied.

The chief question as regards μ is a simple one—within a particular fluid is μ constant? In other words, is viscosity a property of a fluid like inertia which is independent of its motion? If it is, our equations may be useful; if it is not then the introduction of μ into the equations renders them so complex that it is almost hopeless to expect anything from them.

Another question of scarcely less practical importance relates to the character of μ near the bounding surfaces of the fluid. If μ is constant in the fluid, does it change its value near the boundary of the fluid? Is there anything like slipping between the fluid and a solid boundary with which it is in contact?

As regards the answers to these questions the present position is somewhat as follows:—

11. *The Two Viscosities.*

The general experience that the resistance varies as the square of the velocity is an absolute proof that μ is not constant unless a restricted meaning be given to the definition of viscosity, excluding such part of the resistance as may be due, in the way explained by Prof. STOKES,* to internal eddies or cross streams, however insensible these may be, so long as they are not simply molecular motions.

On the other hand in the definite experiments made by COLOMB, and particularly by POISEUILLE, it was found that the resistance was proportional to the velocity, and therefore that μ was absolutely constant—*i.e.*, independent of the velocity.†

To meet this discordance it has been supposed that μ varied with the rate of distortion—*i.e.*, is a function of u/a, but is sensibly constant when u/a is small.‡

To assume this, however, is to neglect POISEUILLE's experiments, in which he found for water the resistance absolutely proportional to the velocity in a tube ·6 mm.

* STOKES's Reprint, vol. i., p. 99.

† Paris Mém. Savans Étrang., tom. 9 (1846), p. 434.

‡ LAMB's "Motion of Fluids," 1879, Art. 180.

diameter up to a velocity of 6 metres per second, which corresponds to a value of $u/a = 20{,}000$.

On the other hand it is found by DARCY* and others in large tubes that the resistance varies as the square of the velocity for values of $\frac{u}{a}$, as low as 1. Thus in a tube of ·6 mm. we have μ constant for all rates of distortion below 20,000, while in a tube of 500 mm. diameter μ is a function of the distortion for all values greater than 1.

It is, therefore, clear that if μ is a function of the distortion it must also be a function of the dimensions of the channels, and in that case μ cannot be considered as a property of the fluid only.

The change in the law of resistance from the simple ratio has, however, been shown by the author to be due to a change in the character of the motion of the fluid from that of direct parallel motion to that of sinuous or eddying motion. †

In the latter case, although the mean motion at any point taken over a sufficient time is parallel to the pipe, it is made up of a succession of motions crossing the pipe in different directions.

The question as to whether, in the case of sinuous motion, μ is to be considered as a function of the velocity or not, depends on whether we regard f as expressing the instantaneous shearing stress at a point, or the mean over a sufficient time. Whether we regard the symbols in the equations of motion as expressing the instantaneous motion or the mean taken over a sufficient time.

If the latter, then μ must be held to include, in addition to the mean stress, the momentum per second parallel to u carried by the cross streams in the negative direction across the surface over which f is measured.

If, however, we regard the motion at each instant, then we must restrict our definition of viscosity by making f the instantaneous value of the intensity of resistance at a point.

This is a quantity which we have, and can have no means of measuring except under circumstances which secure that f is constant for all points over a given surface, and for all instants over a given time.

It thus appears that there are two essentially distinct viscosities in fluids. The one a mechanical viscosity arising from the molar motion of the fluid, the other a physical property of the fluid. It is worth while to point out that, although the conditions under which the first of these, the mechanical viscosity, can exist depend primarily on the physical viscosity, the actual magnitudes of these viscosities are independent, or are only connected in a secondary manner. This is shown by a very striking but little noticed fact. When the motion of the fluid is such that the resistance is as the

* Recherches Expl. Paris, 1852.

† "An Experimental Investigation of the circumstances which determine whether the Motion of Water shall be Direct or Sinuous." Phil. Trans., vol. 174 (1883), p. 935.

square of the velocity, the magnitude of this resistance is sensibly quite independent of the character of the fluid in all respects except that of density. Thus, when in a particular pipe the velocity of oil or treacle is sufficient for the resistance to vary as the square of the velocity, the resistance is practically the same as it would be with water at the same velocity, while the physical viscosity of water is more than one hundred times less.

The answer, then, to the question as to the constancy of μ may be clearly given—μ measures a physical property of the fluid which is independent of its motion. But in this sense μ is the coefficient of instantaneous resistance to distortion at a point moving with the fluid.

This restriction is equivalent to restricting the applications of the equations of motion for a viscous fluid to the cases in which there are no eddies or sinuosities.

This, as shown by the author, is the case in parallel channels so long as the product of the velocity, the width of the channel, and the density of the fluid divided by μ is less than a certain constant value. In a round tube this constant is 1400, or

$$\frac{\mathrm{D}v\rho}{\mu} < 1400.$$

At a temperature of 50°, we have with a foot, as unit of length, for water :—

$$\frac{\mu}{\rho} = 0{\cdot}00001428$$

$$\mathrm{D}v < {\cdot}02,$$

so that if D, the diameter of the channel, be ·001 inch, v would have to be at least 240 feet per second for the resistance to vary other than as the velocity.

As regards the slipping at the boundaries, POISEUILLE'S experiments, as well as those of the author, failed to show a trace of this, although f reached the value of 0·702 lb. per square inch, so that within this limit it may be taken as proved that there is no slipping between any solid surface and water. With other fluids, such as mercury in glass tubes, it is possible that the case may be different; but, as regards oils, the probability seems to be that the limit within which there is no slipping will be much higher than with water.

12. *Experimental determination of the value of μ for olive oil.*

Since the value of μ for water is known for all moderate temperatures, in order to obtain the value for oil it is only necessary to ascertain the relative times taken by the same volumes of oil and water to flow through the same channel, care being taken to make the channel such that there are no eddies and that the energy of motion is small compared with the loss of head.

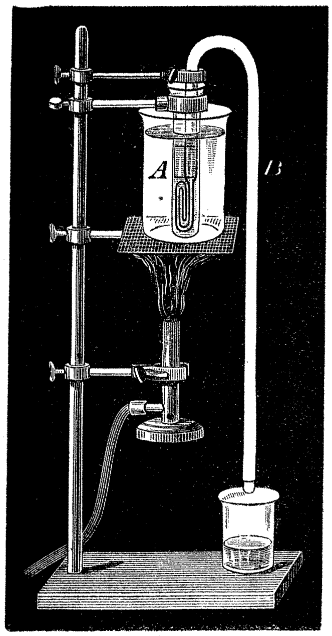
These times are proportional to

$$\frac{\mu}{p}.$$

Where p is the fall of pressure; therefore the times multiplied by the respective falls of pressure are proportional to the viscosities.

The arrangement of apparatus used is shown in fig. 2.

Fig. 2.



The test tube (A) containing the fluid to be tested was fixed in a beaker of water, which was heated and maintained at any required temperature from below.

A syphon (B), made of glass tube $\frac{3}{16}$ inch internal diameter, with the extremity of its short limb drawn down to capillary size for a length of about 6 inches, this six inches being bent up and down so as only to occupy some 2 inches at the bottom of the test tube. The long limb of the syphon extended to about 2 feet below the mean level of the fluid in the test tube. Two marks on the test tube at different levels served to show when a definite volume had been withdrawn.

The syphon used was the same for each set of experiments on oil and water, so that the pressure urging the fluid through the tube was proportioned to the density of the fluids—that is, it was 1·915 as great for oil as water, disregarding the effect of the variation of temperature on volume, which in no case amounted to 1 per cent.

Experiments were first made with water at different temperatures, the times taken for the water to fall from the first mark to the second being carefully noted. The syphon was then dried and replaced and oil substituted for water.

Two sets of similar but entirely different apparatus were used on different occasions, different samples of oil being used. In the first set the experiments on oil were made at temperatures from 95° to 200° Fahr.; in the second set, from 61° to 120° Fahr. In so far as the temperatures overlapped, the viscosities for the two oils agreed to within 4 per cent., but as the law of variation of the viscosity seemed to change rapidly at about 140° Fahr., only the second set have been recorded. These are shown in Table I.

From POISEUILLE'S experiments it is found that, measuring viscosity in pounds on the square inch, for water at a temperature of 61° Fahr.,

$$\mu = 10^{-7} \times 1{\cdot}61.$$

Adopting this value of μ for the experiments on water at 61° Fahr., the other experimental values of μ for water at different temperatures, obtained as being in the ratios of the times, were found to be in very close agreement with those calculated from POISEUILLE'S law for the respective temperatures. This tested the efficiency of the apparatus. It has not been thought necessary to record any experiment on water except at the temperature of 61° Fahr.

The experimental value of μ for oil are in the ratios of the times multiplied by ·915, the specific gravity of oil; these are given as the experimental values of μ in the table. Another column contains the values of μ for oil, calculated from an empirical formula fitted to the experimental values.

This formula was found by comparing the logarithms of the experimental values of μ. It appeared that the differences in these logarithms were nearly proportional to the differences in the corresponding temperatures, or that T being temperature in degrees Fahr.,

$$\log \mu_1 - \log \mu_2 = {\cdot}0096(T_2 - T_1)$$

in degrees Centigrade

$$\log \mu_1 - \log \mu_2 = {\cdot}00535(T_2 - T_1)$$

whence since

$$\begin{aligned} {\cdot}0096 &= {\cdot}0221 \log_{10} e \\ {\cdot}00535 &= {\cdot}0123 \log_{10} e \end{aligned}$$

for degrees Fahrenheit

for degrees Centigrade

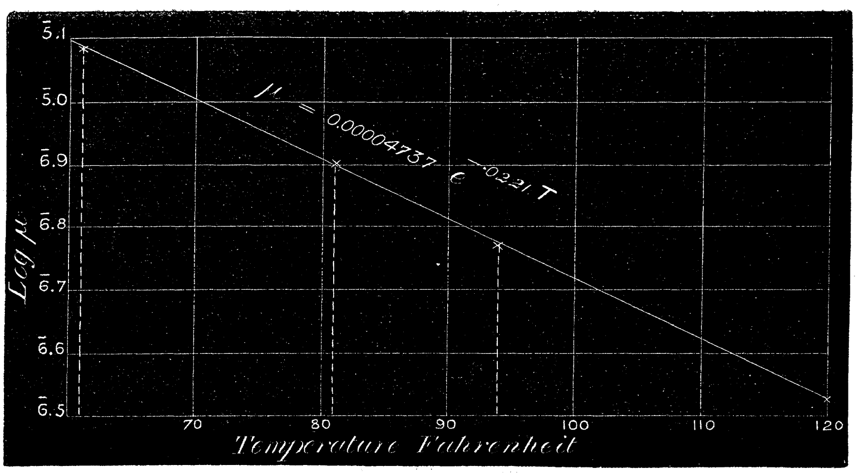
$$\left.\begin{aligned} \frac{\mu_1}{\mu_2} &= e^{-{\cdot}0221(T_1 - T_2)} \\ \frac{\mu_1}{\mu_2} &= e^{-{\cdot}0123(T_1 - T_2)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

This ratio holds well within the experimental accuracy from temperatures ranging from 61° to 120° Fahr. This is shown in the table, and again in fig. 3, in which the ordinates are proportional to $\log \mu$, the abscissæ being proportional to the corresponding temperatures.

TABLE I.—Viscosity of Oil compared with Water: 11 April, 1884.

Number.	Third.	Temperature.		Time seconds.	$\frac{\mu}{107}$ experimental.	log μ experimental.	log μ calculated.	$\frac{\mu}{107}$ calculated.
		Fahrenheit.	Centigrade.					
1	Water. .	60°	15°·5	25	1·640			
2	„ . .	„	„	„	„			
3	„ . .	„	„	„	„			
4	Olive oil .	61	16°	2040	123·00	$\bar{5}$·08990	$\bar{5}$·090133	123·06
5	„ .	81		1350	81·00	$\bar{6}$·90848	$\bar{6}$·89807	79·08
6	„ .	94		1000	60·00	$\bar{6}$·77815	$\bar{6}$·78290	59·34
7	„ .	120		555	33·40	$\bar{6}$·52375	$\bar{6}$·52375	33·40

Fig. 3.



13. *The Comparative Values of μ for Different Fluids and Different Systems of Units.*

The values of μ given by different writers for air and water, are expressed in various units of force and length, so that it is a matter of some trouble to compare them. To facilitate this for the future comparative values are here given. Those for water have been deduced from POISEUILLE'S formula, for air from MAXWELL'S formulæ, and for olive oil from the experiments recorded in the previous article.

The units of length, mass, and time, being respectively the centimetre, gramme, and second, in which case the unit of force is the weight of one 980·5th (g) part of a gramme, expressing temperature in degrees Centigrade by T and putting

$$P^{-1} = 1 + 0{\cdot}0336793T + 0{\cdot}0002209936T^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

for

$$\left.\begin{array}{lll} \text{water} & . \ . & \mu=0{\cdot}0177931\text{P} \\ \text{air} & . \ . \ . & \mu=0{\cdot}0001879(1+0{\cdot}00366\text{T}) \\ \text{olive oil} & . \ . & \mu=3{\cdot}2653\, e^{-{\cdot}0123\text{T}}\text{*} \end{array}\right\} \quad . \ . \ . \ . \ . \ . \quad (4)$$

With the same unit of length, but g grammes as unit of mass and 1 gramme as unit of force, the values of μ are for

$$\left.\begin{array}{lll} \text{water} & . \ . \ . & \mu=0{\cdot}000019792\text{P} \\ \text{air} & . \ . \ . \ . & \mu=0{\cdot}00000019153(1+{\cdot}00366\text{T}) \\ \text{olive oil} & . \ . & \mu=0{\cdot}0033303\, e^{-{\cdot}0123\text{T}} \end{array}\right\} \quad . \ . \ . \ . \ . \quad (5)$$

The units of length and mass being the foot and pound and the temperature in degrees Fahr. for

$$\left.\begin{array}{lll} \text{water} & . \ . \ . & \mu=0{\cdot}0011971\text{P} \\ \text{air} & . \ . \ . \ . & \mu=0{\cdot}000011788(1+{\cdot}0020274\text{T}) \\ \text{olive oil} & . \ . & \mu=0{\cdot}21943\, e^{-{\cdot}0221\text{T}} \end{array}\right\} \quad . \ . \ . \ . \ . \quad (6)$$

With the same unit of length the unit of mass being $g(32{\cdot}1695)$ lbs. and the unit of force 1 lb. for

$$\left.\begin{array}{lll} \text{water} & . \ . \ . & \mu=0{\cdot}000037166\text{P} \\ \text{air} & . \ . \ . \ . & \mu=0{\cdot}00000036645(1+{\cdot}0020274\text{T}) \\ \text{olive oil} & . \ . & \mu=0{\cdot}0068213\, e^{-{\cdot}0221\text{T}} \end{array}\right\} \quad . \ . \ . \ . \ . \quad (7)$$

Taking the unit of length 1 inch and the unit of force 1 lb. for

$$\left.\begin{array}{lll} \text{water} & . \ . \ . & \mu=0{\cdot}000000258105\text{P} \\ \text{air} & . \ . \ . \ . & \mu=0{\cdot}0000000025447(1+{\cdot}0020274\text{T}) \\ \text{olive oil} & . \ . & \mu=0{\cdot}00004737\, e^{-{\cdot}0221\text{T}} \end{array}\right\} \quad . \ . \ . \ . \ . \quad (8)$$

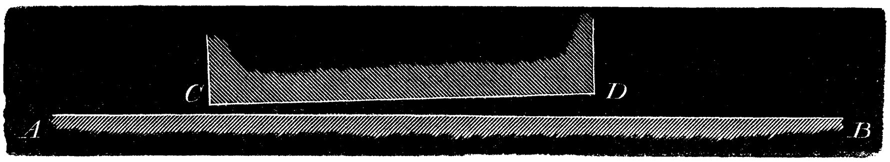
* For olive oil the values of μ have only been tested between the limits of temperature 16° and 49° C. or 61° and 120° Fahr.

SECTION III.—GENERAL VIEW OF THE ACTION OF LUBRICATION.

14. *The case of two nearly parallel Surfaces separated by a viscous Fluid.*

Let AB and CD (fig. 4) be perpendicular sections of the surfaces, CD being of limited but great extent compared with the distance h between the surfaces, both surfaces being of unlimited length in a direction perpendicular to the paper.

Fig. 4.



Case 1. *Parallel Surfaces in Relative Tangential Motion.*—In fig. 5 the surface CD is supposed fixed, while AB moves to the left with a velocity U.

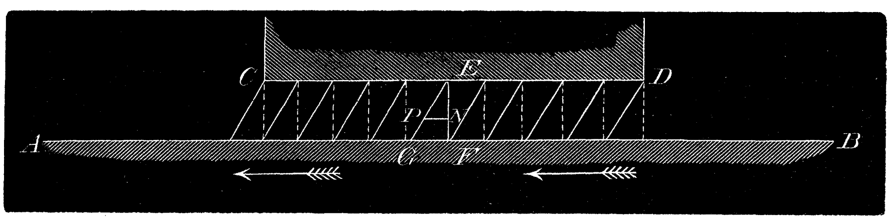
Then by the definition of viscosity (Art. 9) there will be a tangential resistance

$$F=\mu\frac{U}{h},$$

and the tangential motion of the fluid will vary uniformly from U at AB to zero at CD. Thus if FG (fig. 5) be taken to represent U, then PN will represent the velocity in the fluid at P.

The slope of the line EG therefore may be taken to represent the force F, and the direction of the tangential force on either surface is the same as if EG were in tension. The sloping lines therefore represent the condition of motion and stress throughout the film (fig. 5).

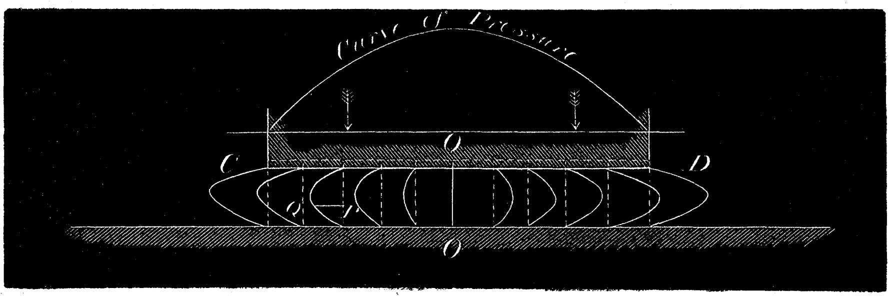
Fig. 5.



Case 2. *Parallel Surfaces approaching with no Tangential Motion.*—The fluid has to be squeezed out between the surfaces, and since there is no motion at the surface, the horizontal velocity outward will be greatest half-way between the surfaces, nothing at O the middle of CD, and greatest at the ends.

If in a certain state of the motion (shown by dotted line, fig. 6) the space between AB and CD be divided into 10 equal parts by vertical lines (fig. 6, dotted figure), and these lines supposed to move with the fluid, they will shortly after assume the positions of the curved lines (fig. 6), in which the areas included between each pair of curved lines is the same as in the dotted figure. In this case, as in Case 1, the distance QP will represent the motion at any point P, and the slope of the lines will represent the tangential forces in the fluid as if the lines were stretched elastic strings. It is at once seen from this that the fluid will be pulled towards the middle of CD by the viscosity as though by the stretched elastic lines, and hence that the pressure will be greatest at O and fall off towards the ends C and D, and would be approximately represented by the curve at the top of the figure.

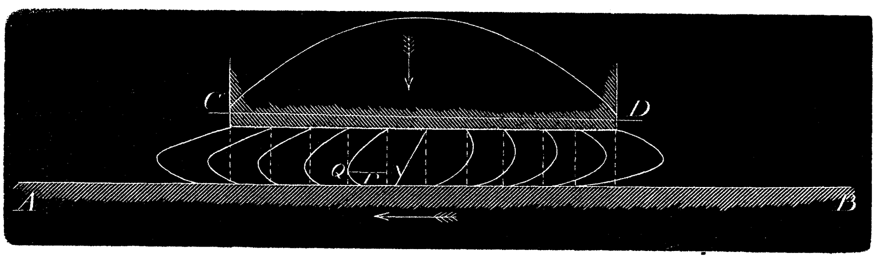
Fig. 6.



Case 3. *Parallel Surfaces approaching with Tangential Motion.*—The lines representing the motions in Cases 1 and 2 may be superimposed by adding the distances PQ in fig. 6 to the distances PN in fig. 5.

The result will be as shown in fig. 7, in which the lines represent in the same way as before the motions and stresses in the fluid where the surfaces are approaching with tangential motion.

Fig. 7.

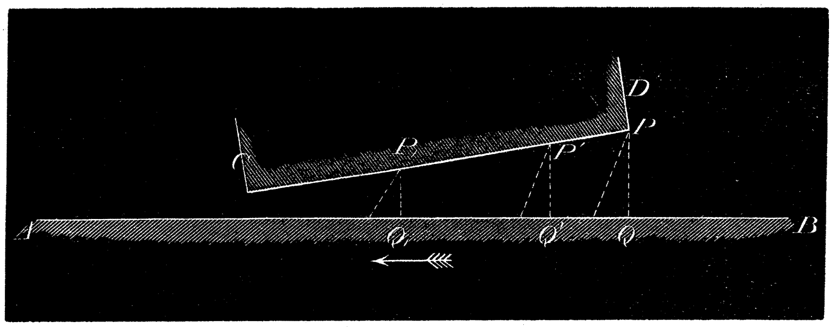


In this case the distribution of pressure over CD is nearly the same as in Case 2, and the mean tangential force will be the same as in Case 1. The distribution of the friction over CD will, however, be different. This is shown by the inclination of the curves at the points where they meet the surface. Thus on CD the slope is greater on the left and less on the right, which shows that the friction will be greater on the left and less on the right than in Case 1. On AB the slope is greater on the right and less on the left, as is also the friction.

Case 4. *Surfaces inclined with Tangential Movement only.*—AB is in motion as in Case 1, and CD is inclined as in fig. 8.

The effect in this case will be nearly the same as in the compound movement (Case 3).

Fig. 8.



For if corresponding to the uniform movement U of AB, the velocity of the fluid varied uniformly from the surface AB to CD, then the quantity carried across any section PQ would be

$$PQ \times \frac{U}{2},$$

and consequently would be proportional to PQ; but the quantities carried across all sections must be the same, as the surfaces do not change their relative distances; therefore there must be a general outflow from any vertical sections PQ, P′Q′ given by

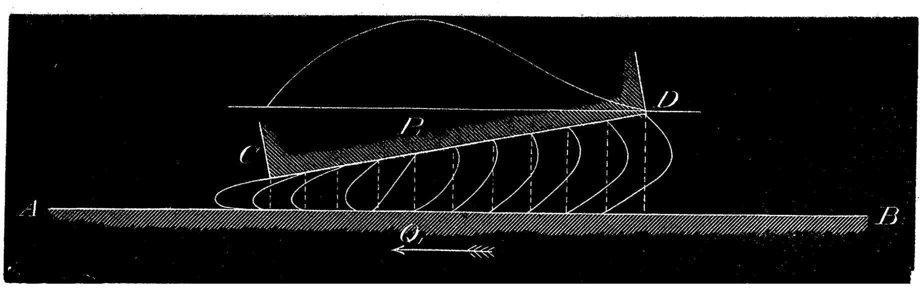
$$\frac{U}{2}(PQ - P'Q').$$

This outflow will take place to the right and left of the section of greatest pressure. Let this be P_1Q_1, then the flow past any other section PQ

$$\frac{U}{2}(PQ - P_1Q_1)$$

to the right or left according as PQ is to the right or left of P_1Q_1. Hence at this section the motion will be one of uniform variation, and to the right and left the lines showing the motion and friction will be nearly as in fig. 7. This is shown in fig. 9.

Fig. 9.



This is the explanation of continuous lubrication.

The pressure of the intervening film of fluid would cause a force tending to separate the surfaces.

The mean line or resultant of this force would act through some point O.

This point O does not necessarily coincide with P_1 the point of maximum pressure.

For equilibrium of the surface AB, O will be in the line of the resultant external force urging the surfaces together, otherwise the surface ACD would change its inclination.

The resultant pressure must also be equal to the resultant external force perpendicular to AB (neglecting the obliquity of CD). If the surfaces were free to approach the pressure would adjust itself to the load, for the nearer the surfaces the greater would be the friction and consequent pressure for the same velocity, so that the surfaces would approach until the pressure balanced the load.

As the distance between the surfaces diminished O would change its position, and therefore, to prevent an alteration of inclination, the surface CD must be constrained so that it could not turn round.

It is to be noticed that continuous lubrication between plane surfaces can only take place with continuous motion in one direction, which is the direction of continuous inclination of the surfaces.

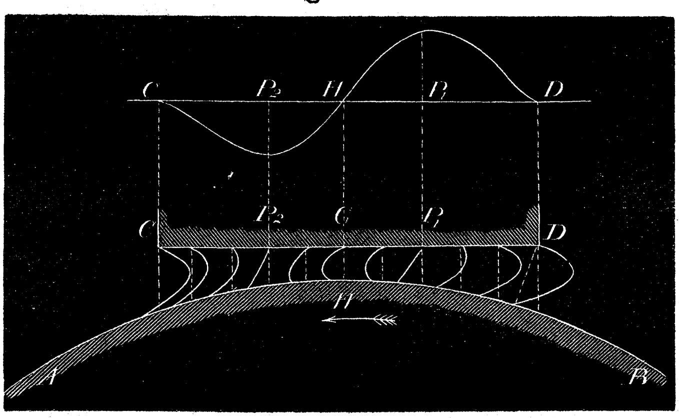
With reciprocating motion, in order that there may be continuous lubrication, the surfaces must be other than plane.

15. *Revolving Cylindrical Surface.*

When the moving surface AB is cylindrical and revolving about its axis, the general motion of the film will differ somewhat from what it is with flat surfaces.

Case 5. *Revolving Motion, CD flat and symmetrically placed.*—The surface velocity of AB may be expressed by U as before. The curves of motion found by the same method as in the previous cases are shown in fig. 10.

Fig. 10.



The curves to the right of GH, the shortest distance between the surfaces, will have the same character as those in fig. 9 to the right of C, at which is also the shortest distance between the surfaces.

On the left of GH the curves will be exactly similar to those on the right, only drawn the other way about, so that they are concave towards a section at P_2 in a similar position on the left to that occupied by P_1 on the right.

This is because a uniformly varying motion would carry a quantity of fluid proportional to the thickness of the stratum from right to left, and thus while it would carry more fluid through the sections towards the right than it would carry across GH, necessitating an outward flow from the position P_1 in both directions, the same motion would carry more fluid away from sections towards C than it would supply past GH, thus necessitating an inward flow towards the position P_2.

Since G is in the middle of CD these two actions, though opposite, will be otherwise symmetrical, and

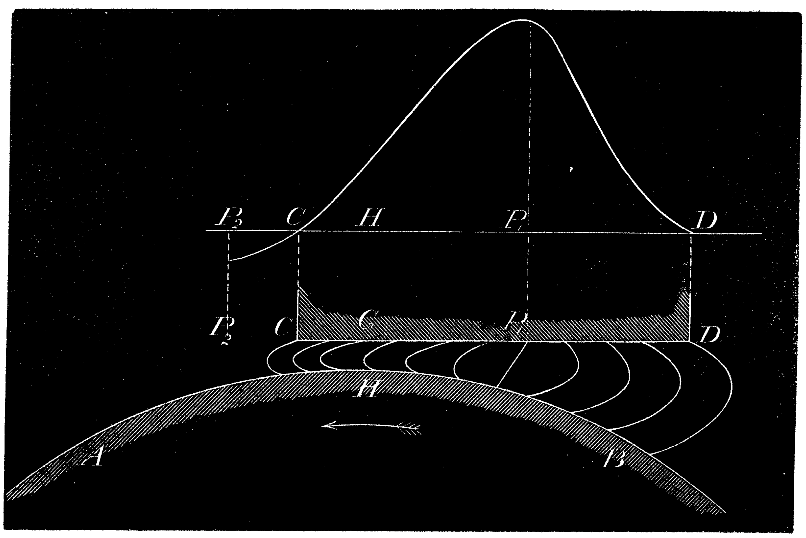
$$P_2G = GP_1.$$

From the convexity of the curves to the section at P_2 it appears that this section would be one of minimum pressure, just as P_1 is of maximum. Of course this is supposing the lubricant under sufficient pressure at C and D to allow of the pressure falling. The curve of pressure would be similar to that at the top of fig. 10, in which C and D are points of equal pressure, P_1HP_2 the singular points in the curve.

Under such conditions the fluid pressure acts to separate the surfaces on the right, but as the pressure is negative on the left the surfaces will be drawn together. So that the total effect will be to produce a turning moment on the surface AB.

Case 6. *The same as Case 5, except that G is not in the middle of CD.*—In this case the curves of motion will be symmetrical on each side of H at equal distances, as shown in fig. 11.

Fig. 11.

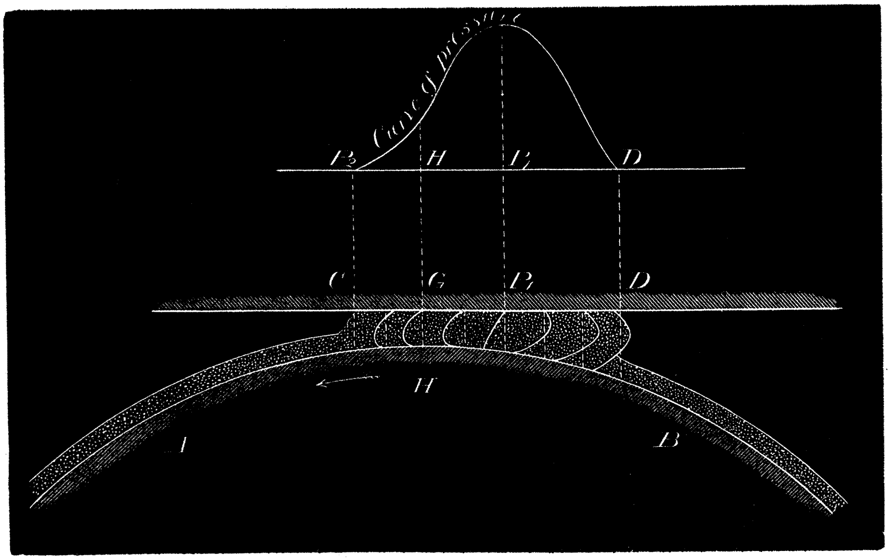


If C lies between H and P_2 the pressure will be altogether positive, as shown by the curve above fig. 11—that is, will tend to separate the surfaces.

16. *The Effect of a Limiting Supply of Lubricating Material.*

In the cases already considered C and D have been the actual limits of the upper surface. If the supply of lubricant is limited C and D may be the extreme points to which the separating film reaches on the upper surface, which may be unlimited, as in fig. 12.

Fig. 12.



Case 7. *Supply of Lubricant Limited.*—If the surface AB be supposed to have been covered with a film of oil, the oil adhering to the surface and moving with it, then the surface CD to have been brought up to a less distance than that occupied by the film of oil, the oil will accumulate as it is brought up by the motion of AB, forming a pad between the surfaces particularly on the side D.

The thickness of the film as it leaves the side C being reduced until the whole surface AB is covered with a film of such thinness that as much leaves at C as is brought up to D, then the condition will be steady.

Putting b for the thickness of the film of oil outside the pad, the quantity of oil brought up to D by the motion of this film will be per second

$$bU,$$

and the quantity which passes the section P_1Q_1, across which the velocity varies uniformly, will be

$$\frac{P_1Q_1U}{2}.$$

Therefore since there is no further accumulation

$$P_1Q_1=2b,$$

also, since $GP_2=GP_1$ (fig. 10, Case 5)

$$P_2Q_2=2b$$

And since the quantity which passes P_2Q_2 will not be sufficient to occupy the larger sections on the left, the fluid will not touch the upper surface to the left of P_2. The limit will therefore be at P_2, the fluid passing away with AB in a film of thickness b.

This is the ordinary case of partial lubrication: AB, the surface of the journal, is covered with a film of oil; CD, the surface of the brass or bearing, is separated from AB by a pad of oil near H, the point of nearest approach.

This pad is under pressure, which is a maximum at P_1, and slopes away to nothing at D and P_2, the extremities of the pad, as is shown by the curve above, fig. 12.

17. *The Relation between Resistance, Load, and Speed for Limited Lubrication.*

In Case 7 a definite quantity of oil must be in the film round the journal, or in the pad between the surfaces. As the surfaces approach the pad will increase and the film diminish, and *vice versâ*. The resistance increases with the length of the pad and with the diminution of the distance between the surfaces. The mean intensity of pressure increases with the length of the pad, and inversely with the thickness of the film, but not in either case in the simple ratio. The total pressure which is equal to the load increases with the intensity of pressure and the length of the pad.

The definite expressions of these relations depend on certain integrations, which have not yet been effected. From the general relations pointed out, it follows that an increase of load will diminish HG and P_1Q_1, and consequently the thickness of the film round the journal, and will increase the length of the pad. It will therefore increase the friction.

Thus with a limited supply of oil the friction will increase with the load in some ratio not precisely determined.

Further, both the friction and the pressure increase in the direct ratio of the speed, provided the distance between the surfaces and the length of the pad remains constant; then, if the load remains constant, the thickness of the film must increase, and the length of the pad diminish with the speed; and both these effects will diminish friction in exactly the same ratio as the reduction of load diminishes friction.

Thus if with a speed U a load W and friction F a certain thickness of oil is maintained, the same will be maintained with a speed MU, a load MW, and the friction will be MF.

How far this increase of friction is to be attributed to the increased velocity, and how far to the increased load, is not yet shown in the theory for this case; but, as has been pointed out, if the load be altered from MW to W, the velocity remaining the same, the friction will be altered from MF in the direction of F. Therefore, with the load constant, it does appear from the theory that the friction will not increase as the first power of the velocity.

There is nothing therefore in this theory contrary to the experience that, with very limited lubrication, the friction is proportional to the load and independent of the velocity, while the theoretical conclusion that the friction, with any particular load and speed, will depend on the supply of oil in the pad, is in strict accordance with Mr. TOWER's conclusion, and with the general disagreement of the coefficients of friction determined in different experiments.

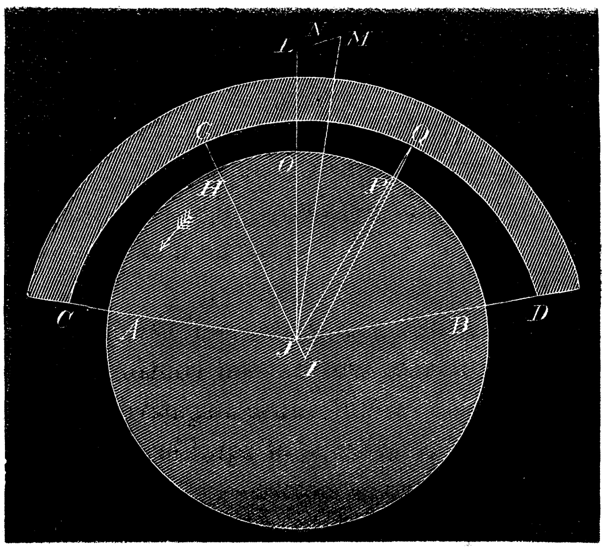
17A. *The Conditions of Equilibrium with Cylindrical Surfaces.*

So far CB has been considered as a flat surface, in which case the equilibrium of CB requires that it should be so far constrained by external forces that it cannot either change its direction or move horizontally.

When AB is a portion of a cylindrical surface, having its axis parallel to that of AB, the only condition of constraint necessary for equilibrium is that CB shall not turn about its axis. This will appear on consideration of the following cases :—

Case 8. *Surfaces Cylindrical and the Supply of Oil Limited.*—Fig. 13 shows the surfaces AB and CD

Fig. 13.



J is the axis of the journal AB.

I is the axis of the brass CD.

JL is the line in which the load acts.

O is the point in which JL meets AB.

$R = JP$.

$R + a = IQ$.

$h = PQ$.

$h_0 = HG$.

The condition for the equilibrium of I is that the resultant of the oil pressure on BC together with friction shall be in the direction OL, and the magnitude of this resultant shall be equal to the load.

As regards the magnitude of this resultant it increases as HG diminishes to a certain limit, *i.e.*, as the surfaces approach, so that in this respect equilibrium is obviously secured, and it is only the direction of the resultant pressure and friction that need be considered.

Since the fluid film is in equilibrium under the forces exerted by the two opposite surfaces these forces must be equal and opposite, so that it is only necessary to consider the forces exerted by AB on the fluid.

From what has been already seen in Cases 6 and 7 it appears that the resultant line of pressure JM always lies on the right or *on* side of GH. The resultant friction clearly acts to the left, so that if JM be taken to represent the resultant pressure and MN the resultant friction, N is to the left of M and JN the resultant of pressure and friction is to the left of JM.

Taking LJ to represent the load, then LN will represent the resultant moving force on CD that is on I. Since H will move in the opposite direction to I, and since the direction of the resultant pressure moves in the same direction as H, the effect of a moving force LN on I will be to move N towards L until they coincide. Thus as long as JM is within the arc covered by the brass a position of equilibrium is possible and the equilibrium will be stable.

So far the condition of equilibrium shows that H will be on the left or off side of the line of load, and this holds whether the supply of oil is abundant or limited; but while with a very limited supply of oil, *i.e.*, a very short oil pad, H must always be in the immediate neighbourhood of O, this is by no means the case as the length of the oil pad increases.

Case 9. *Cylindrical Surfaces in Oil Bath.*—If the supply of oil is sufficient the oil film or pad between the surfaces will extend continuously from the extremities of the brass, unless such extension would cause negative pressure which might lead to discontinuity. In this case the conditions of equilibrium determine the position of H.

The conditions of equilibrium are as before—

1. That the horizontal component of the oil pressure on the brass shall balance the horizontal component of the friction;

2. That the vertical components of the pressure and friction shall balance the load.

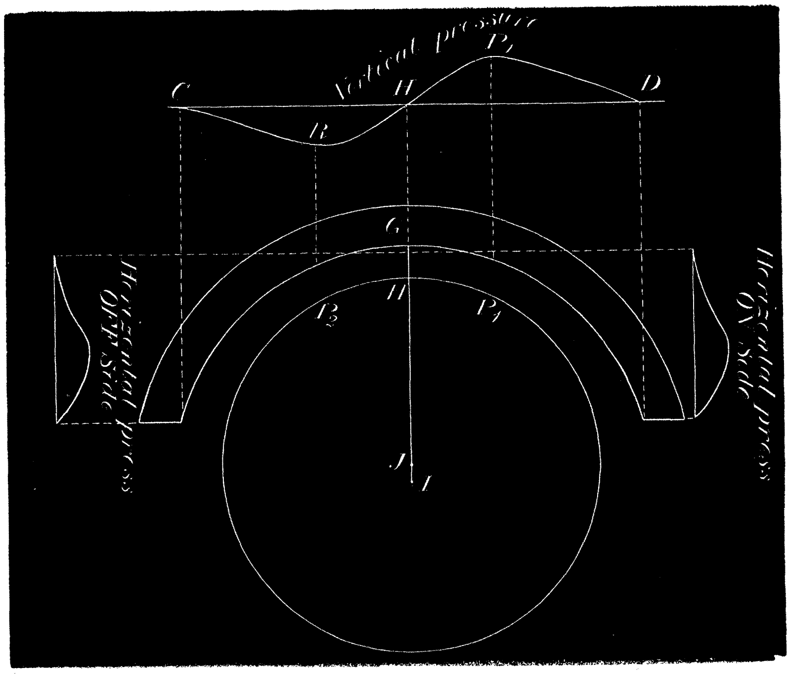
Taking the surface of the brass, as is usual, to embrace nearly half the circumference of the journal and, to commence with, supposing the brass to be unloaded, the movement of H may be traced as the load increases.

When there is no load the conditions of equilibrium are satisfied if the position of H is such that the vertical components of pressure and friction are each zero, and the horizontal components are equal and opposite.

This will be when H is at O (fig. 13); for then, as has been shown, Case 5, the pressure on the left of H will be negative, and will be exactly equal to the pressure at corresponding points on the right, so that the vertical components left and right balance each other. On the other hand the horizontal component of the pressure to the left and right will both act on the brass to the right, and as these will increase as the surfaces approach, the distance JI must be exactly such that these components balance the resultant friction, which by symmetry will be horizontal and acting to the left.

It thus appears that when the brass is unloaded its point of nearest approach will be its middle point. This position, together with the curves of pressure, are shown in fig. 14.

Fig. 14.



As the load increases the positive vertical component on the right of GH must overbalance the negative component on the left. This requires that H should be to the left of O.

It is also necessary that the horizontal components of pressure and friction should balance.

These two conditions determine the position of H and the value of JI.

As the load increases it appears from the exact equations (to be discussed in a subsequent article) that OH reaches a maximum value, which places H nearly, but not quite, at the left extremity of the brass, but leaves JI still small as compared with GH.

For a further increase of the load H moves back again towards O.

In this condition the load has become so great that the friction which remains nearly constant is so small by comparison that it may be neglected, and the condition of equilibrium is that the horizontal component of the pressure is zero, and the vertical component equal to the load.

H continues to recede as the load increases. But when HC becomes greater than HP_2, the pressure between P_2 and C would become negative if the condition did not break down by discontinuity in the oil, which is sure to occur when the pressure falls below that of zero, and then the condition becomes the same as that with a limited supply of oil.

This is important, as it shows that with extreme loads the oil bath comes to be practically the same as that of a limited supply of oil, and hence that the extreme load which the brass would carry would be the same in both cases—as Mr. TOWER has shown it to be.

In all Mr. TOWER's experiments with the oil bath it appears that the conditions were such that H was in retreat as the load increased from C towards O, and that, except in the extreme cases, P_2 had not come up to C.

Figs. 2, 3, 4 (Plate 8), show the exact curves of pressure as calculated by the exact method to be given, for circumstances corresponding very closely with one of Mr. TOWER's experiments, in which he actually measured the pressure of oil at three points in the film. These measured pressures are shown by the crosses.

The result of the calculations for this experiment is to show, what could not indeed be measured, that in Mr. TOWER's experiment the difference in the radii of the brass and journal at 70°, and a load of 100 lbs. per square inch,

$$a = {\cdot}00077$$
$$GH = {\cdot}000375$$
$$\text{(The angle) } OJH = 48^\circ$$

18. *The Wear and Heating of Bearings.*

Before the journal starts the effect of the load will have brought the brass into contact with the journal at O. At starting the surfaces will be in contact, and the initial friction will be between solid surfaces, causing some abrasion.

After motion commences the surfaces gradually separate as the velocity increases, more particularly in the case of the oil bath, in which case at starting the friction will be much the same as with a limited supply of oil.

As the speed increases according to the load, GH approaches, according to the supply of oil, to a, and varies but slightly with any further increase of speed; so that the resistance becomes more nearly proportional to the speed and less affected by the load.

When the condition of steady lubrication has been attained, if the surfaces are completely separated by oil there should be no wear. But if there is wear, as it appears from one cause or another there generally is, it would take place most rapidly where the surfaces are nearest: that is, at GH on the off side of O.

Thus while the motion is in one direction the tendency to wear the surfaces to a fit would be confined to the off side of O.

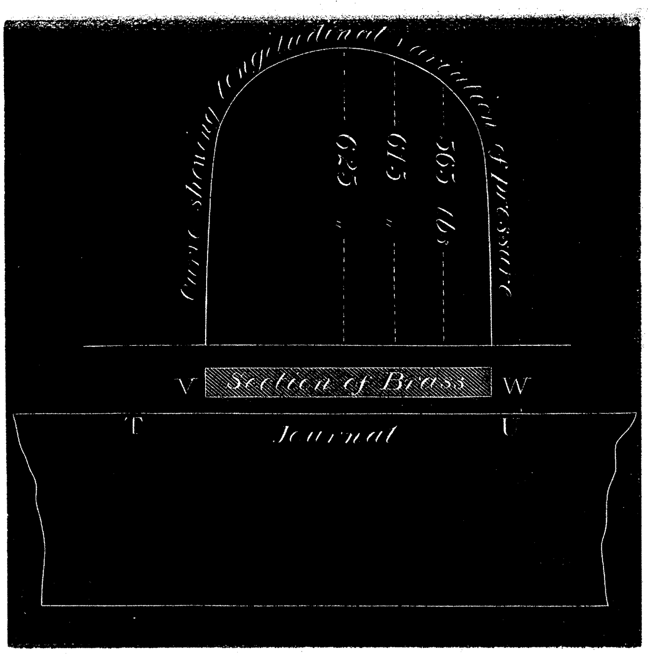
This appears to offer a very simple and well-founded explanation of the important and common circumstance that new surfaces do not behave so well as old ones; and of the circumstance, observed by Mr. TOWER, that in the case of the oil bath, running the journal in one direction does not prepare the brass for carrying a load when the journal is run in the opposite direction. This explanation, however, depends on the effect of misfit in the journal and brass which has yet to be considered.

Case 10. *Approximately cylindrical surfaces of limited length in the direction of the axis of rotation.*—Nothing has so far been said of any possible motion of the fluid perpendicular to the direction of motion and parallel to the axis of the journal. It having been assumed that the surfaces were truly cylindrical and of unlimited length in direction of their axes, and in such case there would be no such flow.

But in practice brasses are necessarily of limited length, so that the oil can escape from the ends of the brass. Such escape will obviously prevent the pressure of the film of oil from reaching its full height for some distance from the ends of the brass and cause it to fall to nothing at the extreme ends.

This was shown by Mr. TOWER, who measured the pressure at several points along the brass in the line through O, and found it to follow a curve similar to that shown in fig. 15, which corresponds to what might be expected from escape at the free ends.

Fig. 15.

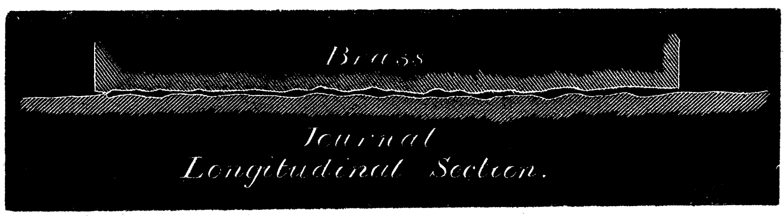


If the surfaces are not strictly parallel in the directions TU and VW, the pressure would be greatest in the narrowest parts, causing axial flow from those into the broader spaces. Hence, if the surfaces were considerably irregular, the lubricant would, by escaping into broader spaces, allow the brass to approach and eventually to touch the journal at the narrowest spaces, and this would be particularly the case near the ends.

As a matter of fact, the general fit of two new surfaces can only be approximate; and how near the approximation is, is a matter of the time and skill spent on preparing, or, as it is called, bedding them. Such bedding as brasses are subject to would not bring them to a condition in which the hills and hollows differed by less than a $\frac{1}{10000}$th part of an inch, so that two such surfaces touching each other on the hills would have spaces as great as a $\frac{1}{5000}$th of an inch between them. This seems a small matter, but not when compared with the mean width of the interval between the brass and the journal which, as will be subsequently shown, was less than $\frac{1}{1000}$th of an inch.

It may be assumed, therefore, that such inequalities generally exist in the surfaces of new brasses and journals. And as the surfaces according to their material and manner of support yield to pressure the brass will close on the journal at its ends, where, owing to the escape of oil, there is no pressure to keep them separate.

Fig. 16.



The section of a new brass and journal taken at GH will therefore be, if sufficiently magnified, as shown in fig. 16, the thickness of the film, instead of being, say, of $\frac{3}{10000}$ths of an inch, varies from 0 to $\frac{5}{10000}$ths, and is less at the ends than at the middle.

In this condition the wear will be at the points of contact, which will be in the neighbourhood of GH on the off side of O (fig. 13), so that if the journal runs in one direction only the surfaces in the neighbourhood of GH (on the *off* side) will be gradually worn to a fit, during which wear the friction will be great and attended with heating, more or less, according to the rate of wear and the obstruction to the escape of heat.

So long, however, as the journal runs in one direction only GH will be on one side (the *off* side) of O, and the wear will be altogether or mainly on this side, according to the distance of H from O.

In the meantime the brass on the *on* side is not similarly worn, so that if the motion of the journal is reversed and the point H transferred to the late *on* side the wear will have to be gone through again.

That this is the true explanation is confirmed if, as seems from Mr. TOWER's report, the heating effect on first reversing the journal was much more evident in the case of the oil bath.

For when the supply of oil is short HG will be very small and H will be close to O. So that the wearing area will probably extend to both sides of O, and thus the brass be partially, if not altogether, prepared for running in the opposite direction.

When the supply of oil is complete, however, as has been shown, H is 50° or 60° from O, unless the load is in excess, so that the wear in the neighbourhood of H on the one side of O would not extend to a point 100° or 120° over to the other side.

Even in the case of a perfectly smooth brass the running of the journal under a sufficient load in one direction should, supposing some wear, according to the theory render the brass less well able to carry the load when running in the opposite direction. For, as has already appeared, the pressure between the journal and brass depends on the radius of curvature of the brass on the *on* side being greater than that of the journal. If then the effect of wear is to diminish the radius of the brass on the *off* side, so that when the motion is reversed the radius of the new *on* side is equal to or less than that of the journal, while the radius of the new *off* side is greater, the oil pressure would not rise. And this is the effect of wear; for as will be definitely shown, the effect of the oil pressure is to increase the radius of curvature of the brass, and as the centre of wear is well on the *off* side, the effect of sufficient wear will be to bring the radius on this side, while the pressure is on, more nearly to that of the journal, so that on the pressure being removed the brass on this side may resume a radius even less than that of the journal.

SECTION IV.—THE EQUATIONS OF HYDRODYNAMICS AS APPLIED TO LUBRICATION.

19. According to the usual method of expressing the stress in a viscous fluid (which is the same as in an elastic solid)* :

$$\left.\begin{aligned} p_{xx} &= -p-\tfrac{2}{3}\mu\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+2\mu\frac{du}{dx} \\ p_{yy} &= -p-\tfrac{2}{3}\mu\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+2\mu\frac{dv}{dy} \\ p_{zz} &= -p-\tfrac{2}{3}\mu\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+2\mu\frac{dw}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\left.\begin{aligned} p_{xy} &= p_{yx}=\mu\left(\frac{dv}{dx}+\frac{du}{dy}\right) \\ p_{yz} &= p_{zy}=\mu\left(\frac{dw}{dy}+\frac{dv}{dz}\right) \\ p_{zx} &= p_{xz}=\mu\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

* STOKE'S "On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids."—Trans. Cambridge Phil. Soc., vol. viii., p. 287. Also reprint, vol. i, p. 84. Also LAMB'S 'Motion of Fluids,' p. 219.

In which the left-hand members are the stresses on plane perpendicular to the first suffix in directions parallel to the second, the first three being the normal stresses, the last six the tangential stresses.

The values of these substituted in the equations of motion

$$\left.\begin{aligned}\rho\frac{\delta u}{\delta t}&=\rho\mathrm{X}+\frac{dp_{xx}}{dn}+\frac{dp_{yx}}{dy}+\frac{dp_{zx}}{dz}\\ \rho\frac{\delta v}{dt}&=\rho\mathrm{Y}+\frac{dp_{xy}}{dx}+\frac{dp_{yx}}{dy}+\frac{dp_{zy}}{dz}\\ \rho\frac{\delta w}{\delta t}&=\rho\mathrm{Z}+\frac{dp_{xz}}{dx}+\frac{dp_{yz}}{dy}+\frac{dp_{zz}}{dz}\\ \frac{\delta \rho}{\delta t}&=\rho\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\end{aligned}\right\} \quad \ldots \ldots \quad (11)$$

give the complete equations of motion for the interior of a viscous fluid.

These equations involve terms severally depending on the inertia and the weight of the fluid, also the variation of stress in the fluid.

In the case of lubrication the spaces between the solid surfaces are so small compared with

$$\frac{\mu}{\mathrm{U}}$$

that the motion of the fluid is shown to be free from eddies as already explained (Art. 11). Also that the forces arising from weight and inertia are altogether small compared with the stresses arising from viscosity.

The equations which result from the substitution from (9) and (10) in the first 3 of (11) may therefore be simplified by the omission of the inertia and gravitation terms, which are the terms involving ρ as a factor.

In the case of oil the remaining terms may still further be simplified by omitting the terms depending on the compressibility of the fluid.

Also if, as is the case, μ is nearly constant the terms involving $d\mu$ may be omitted or considered of secondary importance.

From equations (11) we then have

$$\left.\begin{aligned}\frac{dp}{dx}&=\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)\\ \frac{dp}{dy}&=\mu\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)\\ \frac{dp}{dz}&=\mu\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right)\\ 0&=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\end{aligned}\right\} \quad \ldots \ldots \quad (12)$$

Again, since in the case of lubrication we always have to do with a film of fluid between nearly parallel surfaces, of which the radii of curvature are large compared with the thickness of the film, we may without error disregard any curvature there may be in the surfaces, and put

x for distances measured on one of the surfaces in the direction of relative motion,

z for distances measured on the same surface in the direction perpendicular to relative motion,

y for distances measured everywhere at right angles to the surface.

Then, if the surfaces remain in their original direction, since they are nearly parallel,

v will be small compared with u and w, and the variations of u and w in the directions x and z are small compared with their variations in the direction y.

The equations (12) for the interior of the film then become

$$\left.\begin{aligned} \frac{dp}{dx}&=\mu\frac{d^2u}{dy^2} \\ \frac{dp}{dy}&=0 \\ \frac{dp}{dz}&=\mu\frac{d^2w}{dy^2} \\ 0&=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Equations (10) become

$$\left.\begin{aligned} p_{xy}&=p_{yx}=\mu\frac{du}{dy} \\ p_{yz}&=p_{zy}=\mu\frac{dw}{dy} \\ p_{zx}&=p_{xz}=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

20. The fluid is subject to boundary conditions as regards pressure and velocity. These are—

(1) At the lubricated surfaces the fluid has the velocity of those surfaces ;

(2) At the extremities of the surfaces or film the pressure depends on external conditions.

Thus taking the solid surfaces as $y=0$, $y=h$, and as being limited in the direction x and z by the curve

$$f(xy)=0$$

For boundary conditions

$$\left.\begin{array}{llll} y=0 & u=U_0 & w=0 & v=0 \\ y=h & u=U_1 & w=0 & v=U_1\frac{dh}{dx}+V_1 \\ & f(xy)=0 & p=p_0 & \end{array}\right\} \quad . \quad . \quad . \quad (15)$$

21. Equations (13) may now be integrated, the constants being determined by the conditions (15).

The second of these equations gives p independent of y, so that the first and third are directly integrable, whence

$$\left.\begin{array}{l} u=\frac{1}{2\mu}\frac{dp}{dx}(y-h)y+U_0\frac{h-y}{h}+U_1\frac{y}{h} \\ w=\frac{1}{2\mu}\frac{dp}{dz}(y-h)y \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Differentiating these equations with respect to x and z respectively, and substituting in the last of equations (13)

$$\frac{dv}{dy}=-\frac{1}{2\mu}\left[\frac{d}{dx}\left\{\frac{dp}{dx}(y-h)y\right\}+\frac{d}{dz}\left\{\frac{dp}{dz}(y-h)y\right\}\right]-\frac{d}{dx}\left\{U_0\frac{h-y}{h}+U_1\frac{y}{h}\right\}$$

Integrating from $y=0$ to $y=h$, and substituting from conditions (15)

$$\frac{d}{dx}\left(h^3\frac{dp}{dx}\right)+\frac{d}{dz}\left(h^3\frac{dp}{dz}\right)=6\mu\left\{(U_0+U_1)\frac{dh}{dx}+2V_1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

From equations (16) and (14)

$$\left.\begin{array}{l} p_{yx}=\frac{1}{2}\frac{dp}{dx}(2y-h)+\mu(U_1-U_0)\frac{1}{h} \\ p_{yz}=\frac{1}{2}\frac{dp}{dz}(2y-h) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Putting $f_x f_z$ for the shearing stresses at the solid on the surfaces in the directions x and z respectively, then taking the positive sign when $y=h$, and the negative when $y=0$

$$\left.\begin{array}{l} f_x=\mu(U_1-U_0)\frac{1}{h}\mp\frac{1}{2}\frac{dp}{dx}h \\ f_z=\mp\frac{1}{2}\frac{dp}{dz}h \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Equations 17 and 19 are the general equations of equilibrium for the lubricant between continuous surfaces at a distance h, where h is any continuous function of x and z, and μ is constant.

22. For the further integration of these equations it is necessary to know the exact manner in which x and z enter into h, as well as the function which determines the limit of lubricated surfaces.

These integrations have been effected either completely or approximately for certain cases, which include the chief case of practical lubrication.

Complete integration has been obtained for the case of two parallel circular or elliptical surfaces approaching without tangential motion. This case is interesting from the experiment, treated approximately by STEFAN,* of one surface-plate floating on another in virtue of the separating film of air. It is introduced here, however, as being the most complete as well as the simplest case in which to consider the important effect of normal motion in the action of lubricants. It corresponds with Case 2, section III.

Complete integration is also obtained for two plane surfaces

$$h=h_1\left(1+m\frac{x}{a}\right)$$

between the limits at which $p=\Pi$ (the pressure of the atmosphere)

$$x=0, \qquad x=a,$$

the surfaces being unlimited in the direction of z. This corresponds with Case 4, section III.

For the most important case, that of cylindrical surfaces, approximate integration has been effected for the case of complete lubrication with the surfaces unlimited in the direction of z. Case 9, section III.

SECTION V.—CASES IN WHICH THE EQUATIONS ARE COMPLETELY INTEGRATED.

23. *Two Parallel Plane Surfaces approaching each other, the Surfaces having Elliptical Boundaries.*

Here h is constant over the surfaces, and when

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1, \quad p=\Pi \qquad (20)$$

U_0, U_1 are zero.

Equation (17) becomes

$$\left(\frac{d}{dx^2}+\frac{d}{dz^2}\right)p=\frac{12\mu}{h^3}\frac{dh}{dt} \qquad (21)$$

The solution of which is

$$p=\phi(t)\left\{\frac{x^2}{a^2}+\frac{z^2}{c^2}+C,\right\}+E_1e^{-\frac{x}{a}}\sin\frac{y}{c}+\&c. \qquad (22)$$

* Wien. Sitz. Ber., vol. 69 (1874), p. 713.

Therefore

$$2\phi(t)\left(\frac{1}{a^2}+\frac{1}{c^2}\right)=\frac{12\mu}{h^3}\frac{dh}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and

$$E_1=0, \text{ \&c.}$$

$$C_1=-1+\frac{\Pi}{\phi(t)}$$

$$p-\Pi=\frac{12\mu}{h^3}\cdot\frac{a^2c^2}{a^2+c^2}\left\{\frac{x^2}{a^2}+\frac{z^2}{c^2}-1\right\}\frac{dh}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

From equations (19)

$$\left.\begin{aligned} f_x&=\mp\frac{24\mu}{h^3}\frac{a^2c^2}{a^2+c^2}\cdot x\frac{dh}{dt} \\ f_z&=\mp\frac{24\mu}{h^3}\frac{a^2}{a^2+c^2}\cdot z\frac{dh}{dt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

supposing surfaces horizontal and the upper surface supported solely by the pressure of the fluid. The conditions of equilibrium in this case are obvious by symmetry.

The centre of gravity of the load must be vertically over the centre of the ellipse. Since by symmetry

$$\left.\begin{aligned} &\int_0^c\int_0^{a\sqrt{\left(1-\frac{z^2}{c^2}\right)}} pxdxdz=0 \\ &\int_0^c\int_0^{a\sqrt{\left(1-\frac{z^2}{c^2}\right)}} pzdxdz=0 \\ &\int_0^c\int_0^{a\sqrt{\left(1-\frac{z^2}{c}\right)}} f_xdxdz=0 \\ &\int_0^c\int_0^{a\sqrt{\left(1-\frac{z^2}{c}\right)}} f_zdxdz=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

And

$$W=\int_0^c\int_0^{a\sqrt{1-\frac{z^2}{c}}} p-\Pi dxdy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$=-\frac{3\mu\pi}{h^3}\frac{a^3c^3}{a^2+c^2}\cdot\frac{dh}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Therefore integrating

$$t=\frac{3\mu\pi a^3c^3}{(a^2+c^2)W}\left(\frac{1}{h_2{}^2}-\frac{1}{h_1{}^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

t being the time occupied in falling from h_1 to h_2.

24. *Plane Surfaces of unlimited Length and parallel in the Direction of z.*

The lower surface unlimited in the direction x and moving with a velocity $-U$. The upper surface fixed and extending from $x=0$ to $x=a$. This case corresponds with Case 4, Section III.

The boundary conditions are

$$\left.\begin{array}{lll} \left.\begin{array}{l} x=0 \\ x=a \end{array}\right\} & p=\Pi & \\ y=0 & U_0=-U & \\ y=h & U_1=0 & V_1=0 \\ h=h_0\left(1+m\frac{x}{a}\right) & & \end{array}\right\} \quad (30)$$

p is a function of x only.

And from equation (17), Section. IV., by integration

$$\frac{dp}{dx}=-6\mu U\frac{h-h_1}{h^3} \quad (31)$$

h_1 being the value of h when $x=x_1$ where the pressure is a maximum.

Integrating with respect to x, and putting $p=\Pi$ at the boundaries

$$x_1=\frac{a}{2+m}, \quad (32)$$

$$p=\Pi+6\frac{\mu Ua}{h^2 m}\left\{\frac{1}{1+m\frac{x}{a}}-\frac{1+m}{2+m}\frac{1}{\left(1+m\frac{x}{a}\right)}-\frac{1}{2+m}\right\} \quad (33)$$

$$\int_0^a(p-\Pi)dx=\frac{6\mu Ua^2}{h^2m^2}\left\{\log_e(1+m)-\frac{m}{1+\frac{m}{2}}\right\}. \quad (34)$$

or putting W for the load per unit of breadth, W is a maximum when $m=1\cdot2$ approximately and

$$\frac{W}{a}=\cdot16\mu U\frac{a^2}{h^2}. \quad (35)$$

again, by equation (19)

$$f_x=\mu\frac{U_1}{h} \quad (36)$$

therefore

$$\int_0^a f_x dx=\frac{\mu aU}{h_1 m}\log_e(1+m) \quad (37)$$

and if

$$m=1\cdot2$$

$$F=\cdot6572\,\frac{\mu aU}{h_1} \quad (38)$$

In order to render the application of equations (35) and (38) clear, a particular case may be assumed.

Let

$$\mu=10^{-5}$$

which is the value for olive oil at a temperature of 70° Fahr., the unit of length being the inch, and that of force the lb.

Let

$$U=60 \text{ (inches per sec.)}$$
$$h_1=\cdot 0003.$$

Then from (35), the load in lbs. per square inch of lubricated surface is given by

$$\frac{W}{a}=1070a^2$$

and from 38, the frictional resistance in lbs. per square inch is

$$\frac{F}{a}=1\cdot 31$$

This seems to be about the extreme case of perfect lubrication between plane metal surfaces having what appears to be about the minimum value of h_1.

Section VI.—The Integration of the Equations for the Case of Cylindrical Surfaces.

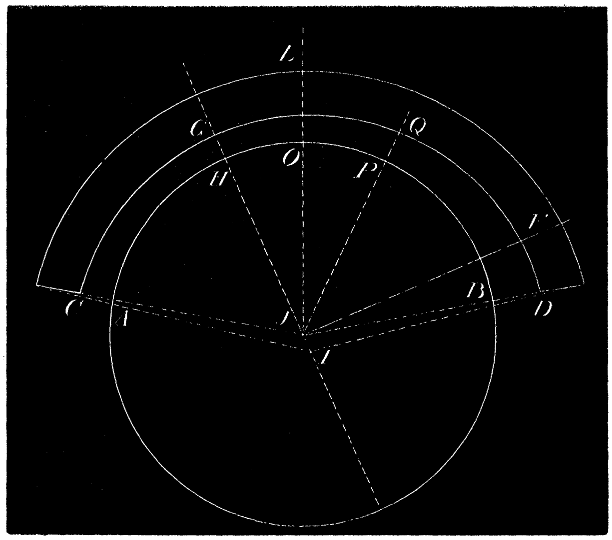
25. *General Adaptation of the Equations.*

Fig. 17 represents a section of two circular cylindrical surfaces at right angles to the axes; as in Art. 17

J is the axis of the journal AB;
I is the axis of the brass CD;
JO is the line of action of the load cutting the brass symmetrically, and

$$\left.\begin{aligned} R&=JP \\ R+a&=IQ \\ h&=PQ \\ h_0&=HG, \text{ the smallest section.} \\ JI&=ca \\ PJO&=\theta \\ GJO&=\frac{\pi}{2}-\phi_0 \\ P_1JO&=\phi_1. \quad P_1 \text{ being the point of maximum pressure.} \end{aligned}\right\} \quad . \quad . \quad . \quad (39)$$

Fig. 17.



Then taking x for distances measured in the direction OA from O on the surface AB, and putting r for the distance of any point from J

$$\left.\begin{array}{l} x=\mathrm{R}\theta \\ y=r-\mathrm{R} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$x_1=\mathrm{R}\phi_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Neglecting quantities of the order $\left(\frac{ca}{\mathrm{R}}\right)^2$

$$h=a\{1+c\sin(\theta-\phi_0)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

For if I be moved up to J, Q moves through a distance ca in the direction JH.

The boundary conditions are such that

(1) all quantities are independent of z
(2) U_0 is constant, U_1 and $\mathrm{V}_1=0$
(3) putting $\theta_0=\mathrm{OJA}$, $\theta_1=\mathrm{OJB}$, whence by symmetry $\theta_0=-\theta_1$

$$\left.\begin{array}{l} \theta=\theta_0 \\ \theta=\theta_1 \end{array}\right\} \ldots p=p_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Putting $-L$ for the effect of the external load and $-M$ for the external moment per unit of length in the direction z, and assuming that there are no external horizontal forces, the conditions of equilibrium for the brass are

$$\int_{\theta_0}^{\theta_1}\{p\sin\theta-f\cos\theta\}d\theta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$\int_{\theta_0}^{\theta_1}\{p\cos\theta+f\sin\theta\}d\theta=\frac{L}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$\int_{\theta_0}^{\theta} fd\theta=\frac{M}{R^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Substituting from equations (40) (41) (42) in equations (17) and (19), Section IV., putting

$$K_1=\frac{6R\mu U_0}{a^2}, \qquad K_2=\frac{\mu U_0}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

and remembering the boundary conditions, these equations become on integration

$$\frac{dp}{d\theta}=\frac{6R\mu U_0c\{\sin(\theta-\phi_0)-\sin(\phi_1-\phi_0)\}}{a^2\{1+c\sin(\theta-\phi_0)\}^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$$f=-\frac{3\mu U_0c\{\sin(\theta-\phi_0)-\sin(\phi_1-\phi_0)\}}{a\{1+c\sin(\theta-\phi)\}^2}$$

$$-\frac{\mu U_0}{\{1+c\sin(\theta-\phi)\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

26. *The Method of Approximate Integration.*

The second numbers of equations (48) and (49) may be expanded so that

$$\left.\begin{aligned}\frac{1}{K_1c}\frac{dp}{d\theta}&=A_0+A_1\sin(\theta-\phi_0)+A_2\cos 2(\theta-\phi)+, \&c.\\ &+A_{2n}\cos 2x(\theta-\phi_0)+A_{2n+1}\sin\{(2x+1)(\theta-\phi)\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (50)$$

$$\left.\begin{aligned}-\frac{1}{K_2}f&=B_0+B_1\sin(\theta-\phi_0)+B_2\cos 2(\theta-\phi)+\ \&c.\\ &+B_{2n}\cos 2x(\theta-\phi)+B_{2n+1}\sin\{(2x+1)(\theta-\phi)\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (51)$$

Putting

$$\chi=\sin(\phi_1-\phi_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$
\left.\begin{aligned}
A_0 &= -\chi-\sum_{r=2}^{r=2\infty}\left\{\frac{(r+1)r^2(r-1)\ldots\frac{r+2}{2}}{2^{r+1}\underline{\left|\frac{r}{2}\right.}}c^{r-1}+\frac{(r+2)(r+1)r(r-1)\ldots\frac{r+2}{2}}{2^{r+1}\underline{\left|\frac{r}{2}\right.}}c^{r}\chi\right\}\\
A_{2n} &= (-1)^{n+1}\left\{\frac{(2n+1)2n}{2^{2n}}c^{2n+1}+\frac{(2n+2)(2n+1)}{2^{2n}}c^{2n}\chi\right.\\
&\qquad\left.+\sum_{r=2n+2}^{r=2\infty}\left[\frac{(r+1)r^2(r-1)\ldots\frac{r+2n+2}{2}}{2^{r}\underline{\left|\frac{r}{2}-n\right.}}c^{r-1}+\frac{(r+2)(r+1)r\ldots\frac{r+2n+2}{2}}{2^{r}\underline{\left|\frac{r}{2}-n\right.}}c^{r}\chi\right]\right\}\\
A_{2n+1} &= (-1)^{n}\left\{\frac{(2n+2)(2n+1)}{2^{2n+1}}c^{2n}+\frac{(2n+3)(2n+2)}{2^{2n+1}}c^{2n+1}\chi\right.\\
&\qquad\left.+\sum_{r=2n+3}^{r=2\infty+1}\left[\frac{(r+1)r^2(r-1)\ldots\frac{r+2n+3}{2}}{2^{r}\underline{\left|\frac{r-2n-1}{2}\right.}}c^{r-1}+\frac{(r+2)(r+1)r\ldots r+2n+2}{2^{r}\underline{\left|\frac{r-2n-1}{2}\right.}}c^{r}\chi\right]\right\}
\end{aligned}\right\}\quad(53)
$$

$$
\left.\begin{aligned}
B_0 &= 1-3c\chi+\sum_{r=2}^{r=2\infty}\left[\frac{(4-3(r+1))r.(r-1)\ldots\frac{r+2}{2}}{2^{r}\underline{\left|\frac{r}{2}\right.}}c^{r}\right.\\
&\qquad\left.-\frac{3(r+1)r(r-1)\ldots\frac{r+1}{2}}{2^{r}\underline{\left|\frac{r}{2}\right.}}c^{r+1}\chi\right]\Bigg\}\\
B_{2n} &= (-1)^{n}\left\{\frac{[4-3(2n+1)]c^{2n}-3(2n+1)c^{2n+1}\chi}{2^{2n-1}}\right.\\
&\qquad\left.+\sum_{r=2n+2}^{r=2\infty}\left[[4-3(r+1)]c^{r}-3(r+1)c^{r+1}\chi\right]\frac{r.(r-1)\ldots\frac{r+2n+2}{2}}{2^{r-1}\underline{\left|\frac{r-2n}{2}\right.}}\right\}\\
B_{2n+1} &= (-1)^{n+1}\left\{\frac{[4-3(2n+2)]c^{2n+1}-3(2n+2)c^{2n+2}\chi}{2^{2n}}\right.\\
&\qquad\left.+\sum_{r=2n+3}^{r=2\infty+3}\left[[4-3(r+1)]c^{r}-3(r+1)c^{r+1}\chi\right]\frac{r.(r+1)\ldots r+2n+3}{2^{r-1}\underline{\left|\frac{r-2n-1}{2}\right.}}\right\}
\end{aligned}\right\}\quad(54)
$$

The coefficients A_0, A_1, &c., B_0, B_1, &c., are thus expanded in a series of ascending powers of c with numerical coefficients which do not converge. It seems, however, that if c is not greater than ·6 the series are themselves convergent, and it is only necessary to go to the tenth or twelfth term, to which extent they have been calculated, and are as follows :—

$$\left.\begin{aligned}
A_0 &= -1{\cdot}5c-3{\cdot}75c^3-6{\cdot}565c^5-9{\cdot}85c^7-13{\cdot}51c^9-17{\cdot}6c^{11} \\
&\qquad -\{1+3c^2+5{\cdot}625c^4+8{\cdot}75c^6+12{\cdot}225c^8+16{\cdot}2c^{10}\}\chi \\
A_1 &= 1+4{\cdot}5c^2+9{\cdot}375c^4+15{\cdot}23c^6+21{\cdot}92c^8+29{\cdot}8c^{10}+38{\cdot}6c^{12} \\
&\qquad +\{3c+7{\cdot}5c^3+13{\cdot}13c^5+19{\cdot}7c^7+27{\cdot}01c^9+35{\cdot}2c^{11}\}\chi \\
A_2 &= 1{\cdot}5c+5c^3+9{\cdot}85c^5+15{\cdot}75c^7+22{\cdot}56c^9+20{\cdot}24c^{11} \\
&\qquad +\{3c^2+7{\cdot}5c^4+13{\cdot}13c^6+19{\cdot}7c^8+27{\cdot}02c^{10}+35{\cdot}2c^{12}\}\chi \\
A_3 &= -1{\cdot}5c^2-4{\cdot}7c^4-9{\cdot}2c^6-14{\cdot}7c^8-21{\cdot}45c^{10} \\
&\qquad -\{2{\cdot}5c^3+6{\cdot}56c^5+11{\cdot}78c^7+18{\cdot}03c^9+25{\cdot}4c^{11}\}\chi \\
A_4 &= -1{\cdot}25c^3-3{\cdot}94c^5-7{\cdot}875c^7-12{\cdot}88c^9-18{\cdot}8c^{11} \\
&\qquad -\{1{\cdot}875c^4+5{\cdot}25c^6+9{\cdot}85c^8+15{\cdot}48c^{10}+22{\cdot}45c^{12}\}\chi \\
A_5 &= {\cdot}939c^4+3{\cdot}07c^6+6{\cdot}33c^8+10{\cdot}68c^{10} \\
&\qquad +\{2{\cdot}63c^5+3{\cdot}94c^7+7{\cdot}76c^9+12{\cdot}6c^{11}\}\chi
\end{aligned}\right\} \quad . \quad (55)$$

$$\left.\begin{aligned}
B_0 &= 1-\{2{\cdot}5c^2+4{\cdot}125c^4+5{\cdot}3125c^6+6{\cdot}54c^8\} \\
&\qquad -\{3c+4{\cdot}5c^3+5{\cdot}625c^5+6{\cdot}562c^7+7{\cdot}63c^9\}\chi \\
B_1 &= 2c+6c^3+9{\cdot}75c^5+11{\cdot}3125c^7 \\
&\qquad +\{6c^2+9c^4+11{\cdot}25c^6+12{\cdot}5c^8\}\chi \\
B_2 &= 2{\cdot}5c^2+5{\cdot}5c^4+7{\cdot}97c^6+10{\cdot}06c^8 \\
&\qquad +\{4{\cdot}5c^3+7{\cdot}5c^5+9{\cdot}85c^7+11{\cdot}8c^9\}\chi \\
B_3 &= -2c^3-4{\cdot}375c^5-6{\cdot}5625c^7-8{\cdot}61c^9 \\
&\qquad -\{3c^4+5{\cdot}625c^6+7{\cdot}873c^8+9{\cdot}84c^{10}\}\chi \\
B_4 &= -1{\cdot}375c^4-3{\cdot}2c^6-5{\cdot}03c^8 \\
&\qquad -\{1{\cdot}875c^5+3{\cdot}94c^7+5{\cdot}09c^9\}\chi
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (56)$$

27. *The Integration of the Equations.*

Integrating equation (50) between the limits θ_0 and θ

$$\begin{aligned}\frac{p-p_0}{K_1 c}=&A_0(\theta-\theta_0)\\ &-A_1\{\cos(\theta-\phi_0)-\cos(\theta_0-\phi_0)\}\\ &+\frac{A_2}{2}\{\sin 2(\theta-\phi_0)-\sin 2(\theta_0-\phi_0)\}\\ &\quad\&c. \qquad \&c.\\ &-\frac{A_{2n+1}}{2n+1}\{\cos[(2n+1)(\theta-\phi_0)]-\cos[(2n+1)(\theta_0-\phi_0)]\}\\ &+\frac{A_{2n}}{2n}\{\sin 2n(\theta-\phi_0)-\sin 2n(\theta_0-\phi_0)\}\ . \qquad (57)\end{aligned}$$

whence putting $\theta=\theta_1$ by condition (43)

$$\begin{aligned}0=&A_0\theta_1-A_1\sin\theta_1\sin\phi_0+\frac{A_2}{2}\sin 2\theta_1\cos 2\phi_0-,\ \&c.\\ &-\frac{A_{2n+1}}{2n+1}\sin[(2n+1)\theta_1]\sin(2n+1)\phi_0\\ &+\frac{A_{2n}}{2n}\sin 2n\theta_1\cos 2n\phi_0\ . \qquad (58)\end{aligned}$$

Putting

$$\begin{aligned}E=&A_1\cos\theta_1\cos\phi_0+\frac{A_2}{2}\cos 2\theta_1\sin 2\phi_0+,\ \&c.\\ &+\frac{A_{2n+1}}{2n+1}\cos[(2n+1)\theta_1]\cos[(2n+1)\phi_0\\ &+\frac{A_{2n}}{2n}\cos 2n\theta_1\sin 2n\phi_0\ . \qquad (59)\end{aligned}$$

whence from equations (57) and (58)

$$\begin{aligned}\frac{p-p_0}{K_1 c}=&E+A_0\theta-A_1\cos(\theta-\phi_0)+\frac{A_2}{2}\sin 2(\theta-\phi_0)\\ &-\frac{A_{2n+1}}{2n+1}\cos[(2n+1)(\theta-\phi_0)]\\ &+\frac{A_{2n}}{2n}\sin 2n(\theta-\phi_0)\ . \qquad (60)\end{aligned}$$

Multiplying equation (60) by $\sin\theta$ and integrating between θ_0 and θ_1, remembering that $\theta_0 = -\theta_1$

$$\int_{\theta_0}^{\theta_1} \frac{p-p_0}{\mathrm{K}c}\sin\theta d\theta = 2\mathrm{A}_0(\sin\theta_1 - \theta_1\cos\theta_1)$$

$$+\mathrm{A}_1\left(\frac{\sin 2\theta_1\sin\phi_0}{2} - \theta_1\sin\phi_0\right)$$

$$+\frac{\mathrm{A}_2}{2}\left(\sin\theta\cos 2\phi - \frac{\sin 3\theta\cos 2\phi}{3}\right)$$

&c. &c.

$$+\frac{\mathrm{A}_{2n+1}}{2n+1}\left\{\frac{\sin(2n+2)\theta_1\sin(2n+1)\phi_0}{2n+2} - \frac{\sin 2n\theta_1\sin(2n+1)\phi_0}{2n}\right\}$$

$$\frac{\mathrm{A}_{2n}}{2n}\left\{\frac{\sin(2n+1)\theta_1\cos 2n\phi}{2n-1} - \frac{\sin(2n+1)\theta\cos 2n\phi}{2n+1}\right\} \quad . \quad . \quad . \quad . \quad (61)$$

Multiplying equation (60) by $\cos\theta$, and integrating from θ_0 to θ_1

$$\int_{\theta_0}^{\theta_1}\frac{p-p_0}{\mathrm{K}c}\cos\theta d\theta = -\mathrm{A}_1\left(\theta_1\cos\phi_0 - \frac{\sin 2\theta_1}{2}\cos\phi_0\right) + \frac{\mathrm{A}_2}{2}\left(\frac{\sin 3\theta_1}{6}\sin 2\phi_0 - \tfrac{3}{2}\sin\theta\sin 2\phi\right) - \&c.$$

$$+\frac{\mathrm{A}_{2n+1}}{2n+1}\left\{\frac{2n+1}{2n+2}\sin(2n+2)\theta\cos(2n+1)\phi - \frac{2n+1}{2n}\sin 2n\theta_1\cos(2n+1)\phi\right\}$$

$$+\frac{\mathrm{A}_{2n}}{2n}\left\{\frac{2n}{2n+1}\sin(2n+1)\theta\sin 2n\phi - \frac{2n}{2n-1}\sin(2n-1)\theta\sin 2n\phi\right\}. \quad . \quad (62)$$

Multiplying equation (51) by $\cos\theta$ and integrating

$$-\int_{\theta_0}^{\theta}\frac{f\cos\theta}{\mathrm{K}_2} = 2\mathrm{B}_0\sin\theta_1 - \mathrm{B}_1\left(\frac{\sin 2\theta_1\sin\phi_0}{2} + \theta_1\sin\phi_0\right)$$

$$+\mathrm{B}_2\left(\frac{\sin 3\theta_1\cos 2\phi_0}{3} + \sin\theta_1\cos 2\phi_0\right)$$

$$-\ \&c.$$

$$+\mathrm{B}_{2n}\left\{\frac{\sin(2n+1)\theta_1\cos 2n\phi_0}{2n+1} + \frac{\sin(2n-1)\theta_1\cos 2n\phi_0}{2n-1}\right\}$$

$$-\mathrm{B}_{2n+1}\left\{\frac{\sin(2n+2)\theta_1\sin(2n+1)\phi_0}{2n+2} + \frac{\sin 2n\theta_1\sin(2n+1)\phi_0}{2n}\right\} \quad . \quad . \quad (63)$$

Multiplying equation (51) by $\sin\theta$ and integrating

$$-\int_{\theta_0}^{\theta_1}\frac{f\sin\theta d\theta}{K_2}=B_1\left(\frac{\sin 2\theta_1\cos\phi_0}{2}-\theta_1\cos\phi_0\right)-B_2\left(\frac{\sin 3\theta_1\sin 2\phi_0}{3}-\sin\theta_1\sin 2\phi_0\right)$$

$$-B_{2n}\left\{\frac{\sin(2n+1)\theta_1\sin 2n\phi_0}{2n+1}-\frac{\sin(2n-1)\theta_1\sin 2n\phi_0}{2n-1}\right\}$$

$$+B_{2n+1}\left\{\frac{\sin(2n+2)\theta_1\cos(2n+1)\phi_0}{2n+2}\ \frac{\sin 2n\theta_1\cos(2n-1)\phi}{2n}\right\}\quad . \quad . \quad . \quad (64)$$

Integrating equation (51)

$$-\int_{\theta_0}^{\theta_1}\frac{fd\theta}{K_2}=2B_0\theta_1-2B_1\sin\theta_1\sin\phi_0+\frac{2B_2}{2}\sin 2\theta_1\cos 2\phi_0$$

$$+\frac{2B_{2n}}{2n}\sin 2n\theta_1\cos 2n\phi_0$$

$$-\frac{2B_{2n+1}}{2n+1}\sin(2n+1)\theta_1\sin(2n+1)\phi_0\quad . \quad . \quad . \quad (65)$$

Substituting from equations (61)-(65) in the equations of equilibrium (44), (45), (46), there results—

From (44)

$$0=2(K_1cA_0+K_2B_0)\sin\theta_1-2K_1cA_0\theta_1\cos\theta_1$$

$$+(K_1cA_1-K_2B_1)\frac{\sin 2\theta_1}{2}\sin\phi_0-(K_1cA_1+K_2B_1)\theta_1\sin\phi_0$$

$$+\ \&c.$$

$$+\Sigma_{n=1}^{n=\infty}\left\{\left(\frac{K_1cA_{2n}}{2n}+K_2B_{2n}\right)\frac{\sin(2n-1)\theta_1\cos 2n\phi_0}{2n-1}\right.$$

$$-\left(\frac{K_1cA_{2n}}{2n}-K_2B_{2n}\right)\frac{\sin(2n+1)\theta_1\cos 2n\phi_0}{2n+1}$$

$$-\left(\frac{K_1cA_{2n+1}}{2n+1}+K_2B_{2n+1}\right)\frac{\sin 2n\theta_1\sin(2n+1)\phi_0}{2n}$$

$$\left.+\left(\frac{K_1cA_{2n+1}}{2n+1}-K_2B_{2n+1}\right)\frac{\sin(2n+1)\theta_1\sin(2n+1)\phi_0}{2n+2}\right\}\quad . \quad . \quad . \quad (66)$$

From (45)

$$\frac{L}{R}=(K_1cA_1-K_2B_1)\left(\frac{\sin 2\theta_1}{2}-\theta_1\right)\cos\phi_0$$

$$+\Sigma_{n=1}^{n=\infty}\left\{(K_1cA_{2n}+K_2B_{2n})\left[\frac{\sin(2n+1)\theta_1}{2n+1}-\frac{\sin(2n-1)\theta_1}{2n-1}\right]\sin 2n\phi_0\right.$$

$$\left.+(K_1cA_{2n+1}-K_2B_{2n+1})\left[\frac{\sin(2n+2)\theta_1}{2n+2}-\frac{\sin 2n\theta_1}{2n}\right]\cos(2n+1)\phi_0\right\}\quad . \quad (67)$$

From (46)

$$-\frac{M}{R^2}=K_2\Big\{2B_0\theta_1-2B_1\sin\theta_1\sin\phi_0+\frac{2B_2}{2}\sin 2\theta_1\cos 2\phi_0$$
$$+\frac{2B_{2n}}{2n}\sin 2n\theta_1\cos 2n\phi_0$$
$$-\frac{2B_{2n+1}}{2n+1}\sin(2n+1)\theta_1\sin(2n+1)\phi_0\Big\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

The equations (66), (67), (68), together with (58), which expresses the boundary conditions as regards pressure, are the integral equations of equilibrium for the fluid between the brass and journal, and hence for the brass.

The quantities involved in these equations are

$$R,\ U,\ M,\ L,\ \mu,\ \theta_1 \quad \text{and} \quad a,\ c,\ \phi_0,\ \phi_1.$$

If, therefore, the former are given, the latter are determined by the solution of these equations.

Section VII.—Solution of the Equations for Cylindrical Surfaces.

28. *c and* $\sqrt{\frac{a}{R}}$ *small compared with Unity.*

In this case equations (55) become

$$\left.\begin{array}{ll} A_0=-\chi & B_0=1 \\ A_1=\ \ 1 & B_1=0 \\ A_2=\ \ 0,\ \&c. & B_2=0,\ \&c. \end{array}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Equation (58) gives

$$0=\chi\theta_1+\sin\theta_1\sin\phi_0\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Equation (66) gives

$$0=(2K_1c\chi-2K_2)\sin\theta_1-2K_1c\chi\theta_1\cos\theta_1-K_1c\frac{\sin 2\theta_1}{2}\sin\phi_0+K_1c\theta_1\sin\phi_0.\quad (71)$$

Equation (67) gives

$$\frac{L}{R}=K_1c\left(\frac{\sin 2\theta_1}{2}-\theta_1\right)\cos\phi_0\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

and equation (68) gives

$$\frac{M}{R^2}=-2K_2\theta_1\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

Also equation (57)

$$p-p_0=K_1c\{\cos\theta_1\cos\phi_0-\chi\theta-\cos(\theta-\phi)\}\quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Eliminating χ between equations (70) and (71)

$$\sin\phi_0=\frac{K_2}{K_1}\frac{2\sin\theta_1}{c\left(\theta-\frac{2\sin^2\theta_1}{\theta_1}+\frac{1}{2}\sin 2\theta\right)} \quad . \quad . \quad . \quad . \quad (75)$$

The equations (74) and (75) suffice to determine a, c, ϕ_1 and ϕ_0 under the conditions $\sqrt{\frac{a}{R}}$ and c small so long as ϕ_0 is not small, in which case the terms retained in the equations become so small that some that have been neglected rise into relative importance.

To commence with let

$$L=0$$

(Cases 5 and 9, Section III.)

Then by (72)

$$\cos\phi_0=0$$

and by (70)

$$\chi=-\frac{\sin\theta_1}{\theta_1}$$

putting for χ its value $\sin(\phi_1-\phi_0)$

$$\cos\phi_1=\frac{\sin\theta_1}{\theta_1} \quad . \quad . \quad . \quad . \quad . \quad (76)$$

Equation (76) gives two equal values of opposite signs for ϕ_1. These correspond to the positions of P_1 and P_2, the points of maximum and minimum pressure.

For the extreme cases

$$\left.\begin{array}{ll}\theta_1=0 & \phi_1=\pm\sqrt{\tfrac{2}{3}}\theta_1 \\ \theta_1=\frac{\pi}{2} & \phi_1=\pm\cos^{-1}\frac{2}{\pi} \\ \theta_1=\pi & \phi=0_1\end{array}\right\} \quad . \quad . \quad . \quad . \quad (77)$$

From equations (73) and (47)

$$a=\frac{2\mu U\theta}{M} \quad . \quad . \quad . \quad . \quad . \quad (78)$$

and from (75)

$$c=\frac{a}{3R}\frac{\sin\theta_1}{\left(\theta_1-\frac{2\sin^2\theta_1}{\theta_1}+\frac{1}{2}\sin 2\theta_1\right)} \quad . \quad . \quad . \quad . \quad (79)$$

When L increases

From (72) and (75)

$$\tan\phi_0=\frac{R}{L}K_2\left(\frac{\sin 2\theta_1}{2}-\theta_1\right)\frac{2\sin\theta_1}{\left(\theta_1-\frac{2\sin^2\theta_1}{\theta_1}+\frac{1}{2}\sin 2\theta_1\right)} \quad . \quad . \quad . \quad (80)$$

Hence as L increases $\tan\phi_0$ diminishes until the approximation fails. This, however, does not happen so long as c is small.

As the load increases from zero, equation (80) shows that G moves away from O towards A.

It also appears from equation (70) that χ and ϕ_1 diminish as ϕ_0 diminishes, and that ϕ_1 is positive as long as the equations hold.

To proceed further it is necessary to retain all terms of the first order of small quantities.

Retaining the first power of c only, equations (55) become

$$\left.\begin{array}{ll} A_0=-1\cdot5c-\chi & B_0=1-3c\chi \\ A_1=\ \ 1+3c\chi & B_1=2c \\ A_2=\ \ 1\cdot5c & B_2=0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (81)$$

From (58)

$$\chi(\theta_1+3c\sin\theta_1\sin\phi_0)=-\sin\theta_1\sin\phi_0-1\cdot5c\theta_1+1\cdot5c\frac{\sin 2\theta_1}{2}\cos 2\phi_0 \quad . \quad . \quad (82)$$

From (66)

$$\begin{aligned} 0=&\{-2K_1c(1\cdot5c+\chi)+2K_2(1-3c\chi)\}\sin\theta_1 \\ &+2K_1c(1\cdot5c+\chi)\theta_1\cos\theta_1 \\ &+\{K_1c(1+3c\chi)-2K_2c\}\frac{\sin 2\theta_1}{2}\sin\phi_0 \\ &-\{K_1c(1+3c\chi)+2K_2c\}\theta_1\sin\phi_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83) \\ &+\tfrac{3}{4}K_1c^2\left(\sin\theta_1-\frac{\sin 3\theta_1}{3}\right)\cos 2\phi_0 \end{aligned}$$

From (67)

$$\begin{aligned} \frac{L}{R}=&-\{K_1c(1+3c\chi)-2K_2c\}\left(\theta_1-\frac{\sin 2\theta_1}{2}\right)\cos\phi_0 \\ &+K_1c\frac{1\cdot5c}{4}(\tfrac{1}{3}\sin 3\theta_1\sin 2\phi_0-\sin\theta_1\sin 2\phi_0) \quad . \quad (84) \end{aligned}$$

From (68)

$$\frac{M}{R^2}=-K_2\{2(1-3c\chi)\theta_1-4c\sin\theta_1\sin\phi_0\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

In the equations (82) to (85) terms have been retained as far as the second power of c, but these terms have very unequal values. As χ and $\sin\phi_0$ diminish c increases, and the products of $c\chi$ or $c\sin\phi_0$ may be regarded as never becoming important and be omitted when multiplied by K_1c or K_2.

Making such omissions and eliminating χ between (82) and (83)

$$\sin\phi_0=\frac{\frac{2K_2}{K_1c}\theta_1\sin\theta_1+c\left\{\frac{3}{4}\theta_1\left(\sin\theta_1-\frac{\sin 3\theta_1}{3}\right)-3(\sin\theta_1-\theta_1\cos\theta_1)\frac{\sin 2\theta_1}{2}\right.}{\theta_1\left(\theta_1-\frac{\sin 2\theta_1}{2}\right)-2(\sin\theta_1-\theta_1\cos\theta_1)\sin\theta_1} \quad . \quad . \quad (86)$$

Equation (86) is a quadratic for c in terms of $\sin\phi_0$, from which it is clear that as c increases from zero ϕ_0 goes through a minimum value when

$$c^2=\frac{2K_2}{K_1}\frac{\theta_1\sin\theta_1}{\frac{3}{4}\theta_1\left(\sin\theta_1-\frac{\sin 3\theta_1}{3}\right)-3(\sin\theta_1-\theta_1\cos\theta_1)\frac{\sin 2\theta_1}{2}} \quad . \quad . \quad . \quad . \quad (87)$$

As the load increases from zero the value of c increases from that of equation (79) to the positive root of (87). As the load continues to increase c further increases, but ϕ_0 again increases, so that, as shown by equation (86), for values of ϕ_0 greater than the minimum there are two loads, two values of c, and two values of χ.

If θ_1 is nearly $\frac{\pi}{2}$ c will be of the order $\sqrt{\frac{a}{2R}}$ when ϕ_0 is small, and $\sin\phi_0$ will be of the order $4c$; so that, so long as $\sqrt{\frac{a}{2R}}$ is sufficiently small, no error has been introduced by the neglect of products and squares of these quantities.

For example

$$\theta_1=1{\cdot}37045 \text{ (78° 31′ 30″ as in Tower's experiments)} \quad . \quad . \quad . \quad . \quad (88)$$

By equation (86)

$$\sin\phi_0=3{\cdot}934c+1{\cdot}9847\frac{a}{cR} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

And by (87) at the minimum value of ϕ_0

$$\left.\begin{aligned} c &= \sqrt{\frac{a}{2R}} \\ \sin\phi_0 &= 5{\cdot}61\sqrt{\frac{a}{R}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Putting $\chi=\sin\phi_1-\sin\phi_0$ equation (82) becomes

$$\sin\phi_1=-{\cdot}16776c+{\cdot}5656\frac{a}{Rc} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

or, when ϕ_0 is a minimum,

$$\sin\phi_1={\cdot}682\sqrt{\frac{a}{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Therefore

$$\chi=4{\cdot}928\sqrt{\frac{a}{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

Equations (84) and (85) give

$$\frac{L}{R}=-1{\cdot}1753K_1c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

$$\frac{M}{R^2}=-2{\cdot}74K_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

whence equations (47)

$$c = \cdot 388 a \frac{\mathrm{L}}{\mathrm{M}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (96)$$

$$\frac{\mu}{a} = - \cdot 3635 \frac{\mathrm{M}}{\mathrm{R}^2 \mathrm{U}_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (97)$$

So long then as $a \frac{\mathrm{L}}{\mathrm{M}}$ is not greater than 0·2, these approximate solutions are sufficiently applicable to any case.

For greater values of $\frac{\mathrm{L}}{\mathrm{M}}$ the solution becomes more difficult, as long however as c is not greater than ·5 the solution can be obtained for any particular value of c.

29. *Further Approximation to the Solution of the Equations for particular Values of c.*

The process here adopted is to assume a value for c. From equations (53) and (54) to find

$$\mathrm{A}_0 = \mathrm{A}_0' + \mathrm{A}_0'' \chi \qquad \mathrm{B}_0 = \mathrm{B}_0' + \mathrm{B}_0'' \chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (98)$$

&c. &c.

where $\mathrm{A}_1' \ \mathrm{A}_1'' \ \mathrm{B}_1' \ \mathrm{B}_1''$ are numerical.

These coefficients are then introduced into equations (58) and (66) which on eliminating χ give one equation for ϕ_0.

The complex manner in which ϕ_0 enters into the equation renders solution difficult except by trial, in which way values of ϕ_0 corresponding to different values of c have been found.

The value of ϕ_0 substituted in equations (58) or (66) gives χ and ϕ_1.

The corresponding values of c_1, ϕ_0 and χ being thus obtained, a complete table might be calculated. This, however, has not been done, as there does not exist sufficient experimental data to render such a table necessary.

What has been done is to obtain ϕ_1 and ϕ_0 for $c = \cdot 5$, θ_1 having the value 1·3704 as in equation (88) and in TOWER's experiments.

The value $c = \cdot 5$ was chosen by a process of trial in order to correspond with the experiments in which Mr. TOWER measured the pressure at different parts of the journal as described in his second report, and as being the greatest value of c for which complete lubrication is certain.

Putting $c = \cdot 5$, equations (43) and (44) give

$$\left.\begin{array}{ll} A_0=-1{\cdot}5351-2{\cdot}3018\chi & B_0=-\ {\cdot}012-2{\cdot}304\chi \\ A_1=\ 3{\cdot}0723+3{\cdot}0721\chi & B_1=\ 2{\cdot}143+2{\cdot}286\chi \\ A_2=\ 1{\cdot}8647+1{\cdot}5360\chi & B_2=\ 1{\cdot}139+\ {\cdot}896\chi \\ A_3=-\ {\cdot}8911-\ {\cdot}6571\chi & B_3=-\ {\cdot}455-\ {\cdot}316\chi \\ A_4=-\ {\cdot}3753-\ {\cdot}2582\chi & B_4=-\ {\cdot}146-\ {\cdot}097\chi \\ A_5=\ {\cdot}1396+\ {\cdot}1343\chi & \end{array}\right\} \quad \ldots \quad (99)$$

Taking

$$\theta_1=1{\cdot}3704 \quad \text{or} \quad 78^\circ\ 31'\ 30''$$

it was found by trial that when

$$\phi_0=48^\circ \quad \ldots \quad (100)$$

and K_2 was neglected (under the circumstances K_2 was about ·0003K_1), equation (58) gave

$$\chi=-{\cdot}82295 \quad \text{or} \quad -\sin 55^\circ\ 22'\ 40''$$

and equation (66) gave

$$\chi=-{\cdot}82274 \quad ,, \quad -\sin 55^\circ\ 21'\ 40'' \quad \ldots \quad (101)$$

The difference ·00021 being in the same direction and about the magnitude of the effect of neglecting K_2.

This solution was therefore sufficiently accurate, and adopting the value of $\phi_1-\phi_0$

$$\phi_1=7^\circ\ 21'\ 40'' \quad \ldots \quad (102)$$

Equations (99) then became

$$\left.\begin{array}{ll} A_0=\ {\cdot}3587 & B_0=\ 1{\cdot}912 \\ A_1=\ {\cdot}5449 & B_1=\ 2{\cdot}263 \\ A_2=\ {\cdot}6010 & B_2=\ {\cdot}303 \\ A_3=-{\cdot}3505 & B_3=-\ {\cdot}195 \\ A_4=-{\cdot}0407 & B_4=\ {\cdot}066 \\ A_5=-{\cdot}0291 & \end{array}\right\} \quad \ldots \quad (103)$$

Substituting the values from equations (100), (102), and (103) in

equation (67)

$$\frac{L}{R}=-1{\cdot}2752K_1c \quad \ldots \quad (104)$$

equation (68)

$$\frac{M}{R^2}=-4{\cdot}7546K_2 \quad \ldots \quad (105)$$

By equation (59)

$$E = -\cdot 25257 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

By equation (60)

$$\frac{p-p_0}{K_1 c} = -\cdot 25257 + \cdot 3587\theta - \cdot 545 \cos(\theta - 48^\circ)$$
$$+ \cdot 3005 \sin 2(\theta - 48^\circ) + \cdot 1168 \cos 3(\theta - 48^\circ)$$
$$- \cdot 0407 \sin 4(\theta - 48^\circ) + \cdot 0058 \cos 5(\theta - 48^\circ) \quad . \quad . \quad . \quad . \quad (107)$$

From equation (107) values of

$$\frac{p-p_0}{Kc}$$

have been found for values of θ differing by 10°, and at certain particular values of θ—

$$\left.\begin{array}{ll} \theta = \pm 29^\circ\ 20'\ 20'' & \text{points at which the pressure was measured} \\ \theta = -\ 7^\circ\ 21'\ 40'' & \text{point of maximum pressure} \\ \theta = -76^\circ\ 38'\ 40'' & \text{point of minimum pressure} \\ \theta = \pm 78^\circ\ 31'\ 30'' & \text{the extremities of the brass} \end{array}\right\} \quad . \quad (108)$$

These are given in Table II.

TABLE II.—The pressure at various points round the bearing.

θ off side.	Arc radius = 1.	$\frac{p-p_0}{-K_1 c}$.	θ on side.	Arc radius = 1.	$\frac{p-p_0}{-K_1 c}$.
° ′ ″			° ′ ″		
0 0 0	0·0	1·0151	0 0 0		1·0151
− 7 21 40	−0·12837	1·0269			
−10 0 0	−0·17453	1·0232	10 0 0	0·17453	0·9331
−20 0 0	−0·34907	0·9412	20 0 0	0·34907	0·8022
−29 20 20	−0·5120	0·7923	29 20 20	0·5120	0·6609
−40 0 0	−0·6981	0·5612	40 0 0	0·6981	0·5003
−50 0 0	−0·8737	0·3349	50 0 0	0·8737	0·3555
−60 0 0	−1·0472	0·1449	60 0 0	1·0472	0·2249
−70 0 0	−1·2217	0·0293	70 0 0	1·2217	0·1002
−76 38 20	−1·3367	−0·001			
−78 31 20	−1·3704	0·0002	78 31 20	1·3704	0·0002

The figs. 2, 3, 4 (Plate 8) represent the results of Table II.

In the curve figure (2) the ordinates are the pressures, and the abscissæ the arcs corresponding to θ.

In the curve figure (3) the ordinates are the same plotted to abscissæ $= R \sin \theta$.

In the curve figure (4) the horizontal ordinates are the same as the vertical ordinates in figs. 2 and 2, and the vertical abscissæ are $= R \cos \theta$.

These theoretical results will be further discussed in Section IX., where they will be compared with Mr. TOWER's experiments.

29A. $c = \cdot 5$ *is the limit to this Method of Integrating.*

In the case considered, in which $\theta = 78° \, 31' \, 20''$, Table II., shows that the pressure towards the extreme off side is just becoming negative. With greater values of c this negative pressure would increase according to the theory.

The possibility of this negative pressure would depend on whether or not the extreme off edge of the brass was completely drowned in the oil bath, a condition not generally fulfilled, and even then it is doubtful to what extent the negative pressure would hold, probably not with certainty below that of the atmosphere.

With an arc of contact anything like that of the case considered it would be necessary, in order to proceed to larger values of c than 5, that the limits between which the equations have been integrated would have to be changed from

$$-\theta_1, \; +\theta_1,$$

to

$$2\left(\phi_0 - \frac{\pi}{2}\right) - \phi_1, \; +\theta_1.$$

This integration has not been attempted, partly because it only applies, in the case of complete lubrication, when the value of $c. > \cdot 5$ renders approximation very laborious, but chiefly because it appears almost obvious that the value of c, which renders the pressure negative at the off extremity of the brass is the largest value of c under which lubrication can be considered certain.

The journal may run with considerably higher values of c, the continuity of the film being maintained by the pressure of the atmosphere, which would be most likely to be the case with high speeds. But although the load which makes $c = \cdot 5$ is not necessarily the limit of carrying power of the journal, it would seem to be the limit of the safe working load, a conclusion which, as will appear on considering Mr. TOWER's experiments, seems to be in accordance with experience.

This concludes the hydro-mechanical theory of lubrication so far as it has been carried in this investigation. There remain, however, physical considerations as to the effect of variations of the speed and load on a and μ which have to be taken into account before applying the theory.

SECTION VIII.—THE EFFECTS OF HEAT AND ELASTICITY.

30.—*μ and a are only to be inferred from the experiments.*

The equations of the last section give directly the friction, the intensity of pressure, and the distance between the cylindrical surfaces, when the velocity, the radii of curvature of the journal and the brass, the length of the brass, and the manner of loading are known (*i.e.*, when U, R, a, θ_1, L, and μ are known); and, further, if M the moment of friction is known, the equations afford the means of determining a when μ is known, or μ when a is known.

The quantities U, R, θ_1, and L are of a nature to be easily determined in any experiment or actual case, and M is easily measured in special experiments, but with a and μ it is different.

By no known means can the difference of radii (a) of the journal and its brass be determined to one ten-thousandth part of an inch, and this would be necessary in order to obtain a precise value of a. As a matter of fact even a rough measurement of a is impossible. To determine a, therefore, it is necessary to know the moment of friction or the distribution of pressure; then if the value of μ be known by experiments such as those described for olive oil (Section II.), a can be deduced from the equations for any particular value of μ. But although the values of μ may have been determined for all temperatures for the particular oil used, and that value chosen which corresponds with the temperature of the oil bath in the experiment, the question still arises whether the oil bath (or wherever the temperature is measured) is at the same temperature as the oil film. Considering the thinness of the film and the rapid conduction of heat by metal surfaces, it seemed at first sight reasonable to assume that there would be no great difference, but when on applying the equations to determine the value of a for one of the journals and brasses used in Mr. TOWER'S experiments, it was found that the different experiments did not give the same values for a, and that the calculated values of a increased much faster with the velocity when the load was constant than with the load when the velocity was constant, it seemed probable that the temperature of the oil film must have varied in a manner unperceived, increasing with the velocity and diminishing the viscosity, which would account for an apparent increase of a.

That a should increase with the load was to be expected, considering that the material of both journal and brass are elastic, and that the loads range up to as much as 600 lbs. per square inch, but there does not appear any reason why a should increase with the velocity unless there is an increase of temperature in the metal. If this occurs, the apparent increase of a would be part real and part due to the unappreciated diminution of μ owing to the rise of temperature.

Until some law of this variation of temperature and of the variation of a with the load is found, the results obtained from the equations, with values of μ corresponding to some measured temperature, such as that of the oil bath or a point in the brass, can only be considered as approximately applicable to actual results. Even so, however, the degree of approximation is not very wide as long as the conditions are such that the journal "runs cool."

But, treated so, the equations fail to show in a satisfactory way what is one of the most important matters connected with lubrication—the circumstances which limit the load which a journal will carry. For, although it may be assumed that the limit is reached when ca, the shortest distance between the surfaces, becomes zero or less than a certain value, yet, according to the equations, assuming a and μ to be constant, the value of c increases directly as U if the load be constant; so that the limiting load should increase with U. But this is not the case for it seems from experiments that at a certain value of U the limiting load is a maximum if it does not diminish for a further increase of U.

Although, therefore, the close agreement of the calculated distribution of the pressure over the bearing with that observed and the approximate agreement of the calculated values of the friction for different speeds and loads, such as result when μ and a are considered constant, seem to afford sufficient verification of the theory, and hence a sufficient insight into the general action of lubrication, without entering into the difficult and somewhat conjectural subject of the effects of heat and elasticity, yet the possibility of obtaining definite evidence as to the circumstances which determine the limits to lubrication, which, not having been experimentally discovered, are a great desideratum in practice, seemed to render it worth while making an attempt to find the laws connecting the velocity and load with a and μ.

As neither the temperature of the oil film or the interval between the surfaces can be measured, the only plan is to infer the law of the variations of these quantities from such complete series of experiments as Mr. TOWER'S. In attempting this, a probable formula with arbitrary constants is first assumed or deduced from theoretical considerations, and then these constants are determined from the experiment and the general agreement tested. In order to determine the actual circumstances on which the constants depend, it is important to obtain the formula from theoretical considerations. This has therefore been done, although these considerations would not be sufficient to establish the formulæ without a close agreement with the experiments.

31. *The Effect of the Load and Velocity to alter the Value of the Difference of Radii of the Brass and Journal,* i.e., *of* a.

The effect of the load is owing to the elasticity of the materials, hence it is probable that the effect will be proportional to the load L. To express this put—

$$a=a_0+m\text{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

The effect of the temperature on a is owing to the different coefficients of expansion of brass and iron. Thus :—

$$a_T - a_0 = (B' - S')\,(T - T_0)R,$$

where B′ and S′ are the respective coefficients of expansion of the bearing and journal. These for brass and iron are :—

$$B' = \cdot 0000111$$
$$S' = \cdot 0000061,$$

therefore putting $T - T_0 = Tm$ (the mean rise of temperature due to friction)

$$a_T = a_0\left(1 + \cdot 000005\frac{R}{a_0}Tm\right)$$
$$= a_0(1 + ETm) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

Putting

$$E \text{ for } \cdot 000005\frac{R}{a_0}$$

If, as seems general, a_o is about ·0005 inch, then with a 4-inch journal

$$E = \cdot 02 \text{ about} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

which is sufficiently large to be important.

32. *The Effect of Speed on the Temperature.*

Putting—

T_0 the temperature of the surroundings and bath..
T_1 the mean temperature of the oil as it is carried out of the film.
T_m the mean rise in temperature of the film due to friction.
Q the volume of the oil carried through per second.
D the density.
S the specific heat.
H the heat generated.
H_1 the heat lost by conduction.
C′ a coefficient of conduction.

taking the inch, lb. and degree Fahr. as units, and 12 J as mechanical equivalent of heat.

$$Q = \frac{Uh_1}{2}$$

$$H = \frac{2R\theta fU}{12J}$$

$$H_1 = 2R\theta C'T_m$$

$$T_1 - T_0 = \frac{H - H_0}{DSQ}.$$

Putting

$$H-H_1=qT_mQ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

where q is a constant depending on the relative values of $T-T_0$ and T_m, also on how far the metal of the journal assists the oil in carrying out heat.

On substituting the values of H_1, H_2, Q, it appears

$$T_m=\frac{fU}{\frac{3JDSqh_1U}{R\theta}+B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112\text{A})$$

$$B=12JC'$$

There does not appear to be any reason to assume any of the quantities in the denominator to be functions of the temperature except h_1. By equation (42)

$$h_1=a\{1+c\sin(\phi_1-\phi_0)\}$$

Equation (112A) may thus have the form

$$T_m=\frac{f}{A(1+ET_m)+\frac{B}{U}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

Where

$$A=3JDSqa_0(1+ml)\{1+c\sin(\phi_1-\phi_0)\} \quad . \quad . \quad . \quad . \quad . \quad (114)$$

This shows that A is a function of the load, increasing as the first power and diminishing as the second power, but the experiment show the effects of these terms are small, and A is constant except for extreme loads.

33. *The Formulæ for Temperature and Friction, and the Interpretation of the Constants.*

From equation (8), Section II.

$$\left.\begin{array}{l}\mu=\mu_0{}^{-C(T-T_0)} \\ C=0221 \text{ (for olive oil)}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

From equations (109) and (110)

$$a=(a_0+ml)\{1+E(T-T_0)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

or approximately since $E(T_1-T_0)$ is small

$$a=(a_0+ml)e^{E(T-T_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

Whence substituting in the equation which results from (51) c being small

$$f=\frac{\mu}{a}\mathrm{U} \qquad \qquad (118)$$

Putting T_x for any particular temperature μ_x a_x corresponding values

$$f=\frac{\mu_x}{a_x+ml}\mathrm{U}e^{-\{\mathrm{C}+\mathrm{E}\}(\mathrm{T}-\mathrm{T}_x)} \qquad \qquad (119)$$

From (113)

$$f=\mathrm{A}\{\mathrm{T}_m+\mathrm{ET}^2_m\}+\frac{\mathrm{B}}{\mathrm{U}}\mathrm{T}_m \qquad \qquad (120)$$

These equations (119) and (120) are independent, and therefore furnish a check upon each other when the constants are known.

Thus substituting the experimental values of U and f in (120) a series of values of T_m are obtained, which when substituted in (119) should give the same value of f.

In these equations the meaning of the constants is as follows:—

C is the rate at which viscosity increases with temperature.

E is the rate at which a increases with temperature, owing to the different expansion of brass and iron.

AU expresses generally the mechanical equivalent of the heat which is carried out of the oil film by the motion of the oil and journal for each degree rise of temperature in the film.

B expresses the mechanical equivalent of the heat conducted away through the brass and journal.

The respective importance of these two coefficients is easily apparent. When the velocities are low but little heat will be carried out, and hence the temperature of the journal depends solely on the value of B. But when the velocities are high, B becomes insignificant compared with AU, and it is A alone which keeps the journal cool.

The value of A may to some extent be inferred from the quantities which enter into it as in equation (114). Thus in the case of Mr. TOWER's experiment, since

$$\mathrm{R}=2 \qquad \theta=1{\cdot}37 \qquad a=\cdot00075 \qquad \mathrm{J}=\cdot772$$
$$\mathrm{D}=0{\cdot}033 \qquad \mathrm{S}=0{\cdot}31 \text{ (for olive oil)}$$

$$\mathrm{A}=\cdot0063q \qquad \qquad (121)$$

It is very difficult to form an estimate of q, but it would seem probable that it has a value not far from 2; and, as will be subsequently shown, in the case of Mr. TOWER's experiments, q is about 3·5 or

$$\mathrm{A}=0{\cdot}0223 \qquad \qquad (122)$$

As B expresses the rate at which the heat generated in the oil film is carried away by conduction through the oil and the surrounding metal, any estimate of its value is very difficult. If we could measure the temperature at the surfaces of the metal, B might be made to depend only on the thickness and conductivity of the oil film. But before heat can escape from the journal or bearing it must pass along intricate metal channels formed by the journal or shaft and its supports; and, on consideration, it appears that in ordinary cases the resistance of such channels would be much greater than the resistance of the oil film itself. For example, in the case of a railway axle, the heat generated must escape either along the journal to the nearest wheel or through the brass and the cast-iron axle box to the outside surface, so that either way it must traverse at least three or four inches of iron. This is about the best arranged class of journals for cooling. In most other cases heat has much further to go before it can escape. However, in every case B will depend on the surrounding conditions and can only be determined by experiment. From the experiments, to be considered in the next section, it appears that

$$B=1 \text{ (about)} \qquad (123)$$

But it is to be noticed that Mr. TOWER has introduced a somewhat abnormal condition by heating the oil bath above the surrounding temperature. For in this way, letting alone the heat generated by friction, there must have been a continual flow of heat from the bath along the journal to the machinery; and, considering the comparatively limited surface of the journal in contact with the hot oil and the large area of section of the journal, it appears unlikely that the temperature of the journal was raised by the bath to anything like the full temperature of the latter, a conclusion which is borne out by Mr. TOWER's experiments with different temperatures in the bath (Table XII., Art. 34), which shows that the temperature of the bath produced a much smaller effect on the friction than would have followed from the known viscosity of oil had the temperature of the oil film corresponded with the temperature of the bath.

Thus the temperature of the film independent of friction is not the temperature of the bath or surrounding objects, and as it is unknown until determined from the experiments, it will be designated as

$$T_x,$$

and the suffix x used to designate the particular value for $T=T_x$ of all those quantities which depend on the temperature as

$$\mu_x \quad a_x$$

If T be the mean temperature of the film,

$$T-T_x=T_m \qquad (123\text{A})$$

where T_m is the rise of temperature due to the film.

33A. *The maximum Load the Journal will carry at any Speed.*

It has already been pointed out that the carrying power of the journal is at its greatest when c is between ·5 and ·6. If, therefore, taking the load constant, c passes through a minimum value as the velocity increases with a constant load, then the load which brings c to a constant value will be a maximum for some particular velocity, and if the particular value of c be that at which the carrying power is greatest, the carrying power will be greatest at that particular speed.

The question whether, according to the theory, journals have a maximum carrying power at any particular speed turns on whether

$$\frac{d\text{L}}{d\text{U}}\ (c \text{ being constant})$$

is zero for any value of U.

This admits of an answer if the values for μ, a, T_x, and equations (119) and (120) hold, for when c is constant $\frac{\text{L}}{\text{K}_1\text{U}}$, $\frac{f}{\text{K}_2\text{U}}$, and A are constant, whence differentiating and substituting it appears that when c is constant

$$\frac{d\text{L}}{d\text{U}}=\frac{\text{L}}{\text{U}}\,\frac{\text{A}+\frac{\text{B}}{\text{U}}+\left(\text{AE}-(3\text{E}+\text{C})\frac{\text{B}}{\text{U}}\right)\text{T}-\text{AE}^2\text{T}^2}{\text{A}+\frac{\text{B}}{\text{U}}+\left\{(3\text{E}+\text{C})\text{A}+(\text{E}+\text{C})\frac{\text{B}}{\text{U}}\right\}\text{T}+\text{AE}(\text{E}+\text{C})\text{T}^2} \quad . \quad . \quad . \quad . \quad (123\text{B})$$

where U is to be taken positive, and T increases as U increases. This shows that $\frac{d\text{L}}{d\text{U}}$, for a constant value of c, changes sign for some value of U if T continues to increase with U.

Hence, according to the theory, the values of L, which make c constant as U increases, approach a maximum value as U increases, and since this value, when c is about ·5, represents the carrying power of the journal, this approaches a maximum as U increases.

Section IX.—Application of the Equations to Mr. Tower's Experiments.

34. *References to* Mr. Tower's *Reports.*

From the experiments described in Mr. Tower's Reports I. and II., in the Minutes of the Institution of Mechanical Engineers, 1884, the journal had a diameter of 4 inches, and the chord of the arc covered by the brass was 3·92 inches, the length of the brass being 6 inches.

The loads on the brass in lbs., divided by 24, are called the *nominal load per square inch.*

The moments of friction in inches and lbs., divided by $24 \times R$, are called the nominal friction per square inch.

These nominal loads, and nominal frictions, with the number of revolutions from 100 to 450, at which they were taken, are arranged in tables for each kind of oil used, and also for the same oil at different temperatures of the bath.

All the tables relative to the oil bath in the first report refer to the same brass and journal. And with this brass, to be here called No. 1, no definite measurements of the actual pressure were made.

The second report contains the account of the pressure measurements, but it is to be noticed that these were made with a new brass, here called No. 2, and that the only friction measurements recorded with this brass are three made at velocities five times less than the smallest velocity used in the case of brass No. 1.

It thus happens that while by the application of the foregoing theory to the friction experiments on brass No. 1 a value is obtained for a, the difference in radii of brass No. 1 and the journal; and from the pressure experiments on brass No. 2 a value is obtained for a in the case of brass No. 2; since these are different brasses there is no means of checking the one estimate against the other.

The following tables extracted from Mr. TOWER's reports are those to which reference has chiefly to be made.

The first of these extracts is a portion of Table I. in Mr. TOWER's first report; this related to olive oil, but corresponds very closely with the results for lard oil also, although not quite so close for mineral oil.

The second extract is Table IX. in the first report; this shows the effect of temperature on the friction of the journal with lard oil.

The third extract is from the second report, being Table XII., representing the oil pressure at different parts of the bearing as measured with brass No. 2.

FROM Table I. (Mr. TOWER's 1st Report, Brass No. 1.) Bath of Olive Oil. Temperature, 90 deg. Fahr., 4-in. journal, 6-in. long. Chord of arc of contact = 3·92 in.

Nominal load lbs. per square inch. Total load divided by 4 × 6.	Nominal friction per square inch of bearing.							
	100 rev. 105 ft. per min.	150 rev. 157 ft. per min.	200 rev. 209 ft. per min.	250 rev. 262 ft. per min.	300 rev. 314 ft. per min.	350 rev. 360 ft. per min.	400 rev. 419 ft. per min.	450 rev. 471 ft. per min.
520	..	·416	·520	·624	·675	·728	·779	·883
468	..	·514	·607	·654	·701	·794	·841	·935
415	..	·498	·580	·622	·705	·787	·870	·995
363	..	·472	·580	·616	·689	·725	·798	·907
310	..	·464	·526	·588	·650	·680	·742	·835
258	·361	·438	·515	·592	·644	·669	·747	·798
205	·368	·43	·512	·572	·613	·675	·736	·818
153	·351	·458	·535	·611	·672	·718	·764	·871
100	·36	·45	·555	·63	·69	·77	·82	·89

FROM Mr. TOWER'S 1st Report, Brass No. 1.

TABLE IX. Bath of Lard Oil. Variation of Friction with Temperature. Nominal Load 100 lbs. per square inch.

Temperature. Fahr.	Nominal friction per square inch of bearing.							
	100 rev. per min.	150 rev. per min.	200 rev. per min.	250 rev. per min.	300 rev. per min.	350 rev. per min.	400 rev. per min.	450 rev. per min.
120°	·24	·29	·35	·40	·44	·47	·51	·54
110	·26	·32	·39	·44	·50	·55	·59	·64
100	·29	·37	·45	·51	·58	·65	·71	·77
90	·34	·43	·52	·60	·69	·77	·85	·93
80	·4	·52	·63	·73	·83	·93	1·02	1·12
70	·48	·65	·8	·92	1·03	1·15	1·24	1·33
60	·59	·84	1·03	1·19	1·30	1·4	1·48	1·56

FROM Mr. TOWER'S Second Report, Brass No. 2. Heavy Mineral Oil. Nominal Load 333 lbs. per square inch. Number of revolutions 150 per minute. Temperature, 90°.

TABLE XII. Oil pressure at differents points of a bearing.

Longitudinal planes	on	centre	off
Pressure per square inch	lb.	lb.	lb.
Transverse plane, middle	370	625	500
Transverse plane, No. 1	355	615	485
Transverse plane, No. 2	310	565	430

The points at which the pressure was measured were at the intersections of six planes, three parallel vertical longitudinal planes parallel to the axis of the journal, one through the axis, and the other two, one on the *on* and the other on the *off* side, both of them at a distance of ·975 inch from that through the axis, three transverse planes, one in the middle of the journal, the other two respectively at a distance of one and two inches on the same side of that through the middle.

In referring to these experimental results in the subsequent articles,

The nominal load per square inch is expressed by L′;

The number of revolutions per minute by N′;

The nominal friction by f';

The effect of the fall of pressure at the ends of the journal on the mean pressure is expressed by $\frac{1}{s}$, thus—

n is a coefficient depending on the way in which the journal fits the shaft.

$$\left.\begin{aligned} L' &= \frac{L}{4s} \\ f' &= \frac{nM}{4R} \\ N' &= -\frac{U \times 60}{2\pi R} \\ s &= 1{\cdot}21 \end{aligned}\right\} \quad \ldots \ldots \quad (124)$$

35. *The Effect of Necking the Journal.*

The expression (124) for f' assumes that the journal was not necked into the shaft. From Mr. TOWER's reports it does not appear whether or not the brass was fitted into a neck on the shaft; but since there is no mention of such necking, the theory is applied on the supposition that there was not.

If there were, the friction at the ends of the brass would increase the moment of friction. Put b for the depth of the neck and a' for the thickness of the oil film at the ends, then the moment of resistance of these ends would be—

$$\frac{\mu}{a'} \times \frac{N\pi}{30}\left(R+\frac{b}{2}\right)^3 b.$$

Hence if M be the moment of friction of the cylindrical portion of the journal only

$$24f' - 2\frac{\mu}{a}\frac{\pi N}{30}\, 6\left(R+\frac{b}{2}\right)^3 = 6\frac{M}{R} \quad \ldots \ldots \quad (125)$$

And from equation (97)—

$$f' = \frac{\pi N}{360}\frac{\mu}{a}\left\{\frac{ab\left(R+\frac{b}{2}\right)^3}{a'R} + 8{\cdot}25R^2\right\} \quad \ldots \ldots \quad (126)$$

For example, a 5-inch shaft necked down to a 4-inch journal would give $b = {\cdot}5$ inch. Whence, assuming

$$\left.\begin{aligned} a' &= {\cdot}0005 \\ \text{and} \quad \frac{a}{a'} &= 2 \end{aligned}\right\} \quad \ldots \ldots \quad (127)$$

the relative friction of the ends to that of the journal would be 11·36 to 31·00, or

28 per cent. of the friction; and the values of a, calculated on the assumption of no necking, would have to be increased in the ratio $n=1{\cdot}33$.

Even if there is no necking the value of a will probably not be the same all along the journal, in which case the values of a^2 and a in K_1 and K_2 will be means, and then the square of the mean will be less than the mean of the squares, so that n will probably have a value greater than unity, although there may be no necking of the shaft.

36. *A first Approximation to the Difference in the Radii of the Journal and Brass No. 1.*

The recorded temperature in Mr. TOWER's Table I. is 90° Fahr. Accepting this, and taking the value of μ, equation (8), Section II.,

$$\mu_{90}=10^{-6}\times 6{\cdot}81 \qquad (128)$$

By equation (97)—

$$\frac{n\mu}{a}=\frac{-n\mathrm{M}}{2{\cdot}75\mathrm{R}^2\mathrm{U}} \qquad (129)$$

Since $\mathrm{R}=2$ and $\mathrm{U}=-\frac{4\pi\mathrm{N}}{60}$

$$\frac{n\mu}{a}=-\frac{{\cdot}72f'}{\mathrm{U}}$$

$$=3{\cdot}46\frac{f'}{\mathrm{N}} \qquad (130)$$

Whence substituting from equation (128) for μ_{90}

$$\frac{a}{n}=10^{-6}\times 1{\cdot}97\frac{\mathrm{N}}{f'} \qquad (131)$$

and from the tabular Nos. for $\mathrm{L}'=100$.

$$\left.\begin{aligned} &\mathrm{N}=100, \quad f'={\cdot}36 \\ &\qquad \frac{a}{n}=10^{-4}\times 5{\cdot}5 \text{ (inch)} \\ &\mathrm{N}=480, \quad f'={\cdot}89 \\ &\qquad \frac{a}{n}=10^{-4}\times 10 \text{ (inch)} \end{aligned}\right\} \qquad (132)$$

These are the extreme cases; for intermediate velocities intermediate values of $\frac{a}{n}$ are found.

In order to be sure that these are the values of $\frac{a}{n}$, which result from the application of the equations, it is necessary (since the approximate equations only have been used) to see that the squares of c may be neglected.

Substituting from equation (124) in (96)

$$c = \cdot 388 na \frac{L' s}{f' R}$$

which for L′=100, N=100, gives

$$c = \cdot 033 n^2 \qquad (133)$$

and for L′=100, N=450,

$$c = \cdot 065 n^2$$

So that the approximations hold, and, as already stated in Art. 30, this considerable increase in the value of a with the load constant suggests that the temperature of the film was not really 90°. And as this point has been considered in the last section the equations of that section may be at once used to determine the law of this temperature, after which the values of $\frac{a}{n}$ may be determined with precision.

37. *The Rise in Temperature of the Film owing to Friction.*

In order to determine the values of E, A, and B in equations (119) and (120), by substituting in these equations corresponding values of N and f' for L′=100, the tabular values of f' were somewhat rectified by plotting and drawing the curve N, f'. These corrected values are in the second row, Table III.

From these values, and the corresponding values of N, it was then found by trial that the equations (119) and (120) respectively

$$f = \frac{\mu_x}{a_x + mL} U e^{-(C+E)(T-T_x)}$$

$$f = A\{T_m + ET_m^2\} + \frac{B}{U}(T_m)$$

$$c = \cdot 0221$$

are approximately satisfied for values of $T - T_x = T_m$, if

$$\frac{\mu_x}{a_x + m \times 400} = \frac{\cdot 01345}{n} \qquad (134)$$

$$\left.\begin{aligned} A &= \cdot 022308 \\ E &= \cdot 0222 \\ B &= \cdot 95914 \end{aligned}\right\} \qquad (135)$$

Which seemed to agree very well with the reasoning Section VIII. With these values of the constants the values of $T-T_x$ were then calculated from equation (120), and are given in the third row in Table III. These temperatures were then substituted in equation (119), and the corresponding values of f' calculated, these are given in the fourth row, Table III.

TABLE III.—Rise of Temperature in the Film of Oil caused by Friction, calculated by Equation (120) from Experiments with a Nominal Load 100 lbs. (see Table I., TOWER, Art. 34).

Nominal friction per square inch, as calculated by equation (119) from the rise of temperature.										
Nominal load per square inch $L'=100$ lbs.	N	Revolutions per minute	100	150	200	250	300	350	400	450
	f'	Nominal friction per square inch for olive oil — Table I., TOWER	·36	·45	·54	·63	·69	·77	·82	·89
		Nominal friction per square inch for olive oil — Corrected to a curve . . .	·33	·55	·55	·63	·705	·768	·83	·89
	$T-T_0$ Fahr.	Rise of temperature by equation (120)	3·45	5·83	8·13	10·02	11·77	13·26	14·48	15·37
	f'_1	Nominal friction per square inch calculated by equation (119) assuming c small	·336	·453	·546	·628	·697	·76	·823	·89

The agreement between these calculated values of f' and the experimental values is very close; and it may be noticed that a very small variation in any of A, B and E makes a comparatively large difference in some one or other of the calculated values of f', or, in other words, these are the only positive values of these quantities which satisfy the equations.

The only difference between the experimental and calculated values of f which is not explainable as experimental error, is for the lowest speed at which the experimental value of f' is 6·7 per cent. too large. This is important as it is in accordance with what might be the result of neglecting c^2, since at that speed c is becoming too large to be neglected, and taking c^2 into account the calculated value of f agrees very closely with the experimental.

38. *The Actual Temperature of the Film.*

Having found the approximate values of T_m, the rise of temperature, owing to friction, it remains to find T_x, the temperature of the film, the rise due to friction, so that

$$T_x+T_m=\text{temperature of film.}$$

This is found from Mr. TOWER'S Table XI. (see Art. 34).

Putting

T_B for temperature of the bath

T_0 for temperature of surrounding objects

and assuming

$$T_x+T_m=Z(T_B-T_0)+T_m+T_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

From equations (119) and (130)

$$f' = \frac{N}{3 \cdot 46} \cdot \frac{n\mu_0}{a_0 + mL} e^{-\cdot 0443\{Z(T_B - T_0) + T_m\}}$$

whence

$$\log f' = -\cdot 0443\{Z(T_B - T_0) + T_m\} \log e + \log\left(\frac{N}{3 \cdot 46} \frac{\mu_0}{a_0 + mL}\right). \quad . \quad . \quad (137)$$

In Table IX. (TOWER) the values of f' are given for the same values of N′ and L′ corresponding to different values of T_B.

Substituting corresponding values of f' and T_B in equation (137), and subtracting the resulting equations, we have an equation in which the only unknown quantities are Z and the differences of T_m.

The values of f' being known, the values of T_m are obtained from equations (120), (134), and (125), and substituting these, the equations resulting from (137) give the values of Z. Thus from Table IX. (TOWER)

$$L' = 100$$
$$U = 100$$

$T_B = 60$	$f = \cdot 59$	$T_m = 5 \cdot 9$
$T_B = 70$	$f = \cdot 48$	$T_m = 4 \cdot 8$

From (137)

$$Z + \frac{5 \cdot 9 - 4 \cdot 8}{10} = \frac{\log \cdot 59 - \log \cdot 48}{\cdot 0443 \times 10 \times \log e}.$$

Therefore

$$Z = 35 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

From this value of Z the values of f' corresponding to those in TOWER's Table IX. have been calculated and agree well with the experimental values.

The smallest temperature of the oil bath recorded in TOWER's Table IX. is 60° Fahr., therefore it is assumed that this was the normal temperature, whence

$$T_x = \cdot 35(T_B - 60) + 60 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

Hence it is concluded that the actual temperature of the oil film in all the experiments with the bath, at a temperature of 90° Fahr., is given by

$$T = 70 \cdot 5 + T_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

By the formula for μ since $T_x = 70 \cdot 5$

$$\mu_x = \cdot 00004737 e^{-\cdot 0021 \times 70 \cdot 5}$$
$$= \cdot 000009974 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$
$$= \cdot 00001 \text{ (approximately)}$$

and since, by equation (134), when L′=100

$$a_x = \frac{n\mu_x}{\cdot 01345}$$
$$= \cdot 0007413n \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$
$$= \cdot 00074n \text{ (approximately)}.$$

This is the value of a_x with a load of 100 lbs. per square inch.

39. *The Variation of a with the Load.*

All Mr. Tower's experiments, when the loads are moderate and the velocities high, show a diminution of resistance with an increased load.

Since c increases with the load and the friction increases as c increases, a and μ being constant, the diminution of friction with increased loads shows either that the load increases the temperature of the film and so diminishes the viscosity, or increases the radius of curvature of the brass as compared with that of the journal, *i.e.*, increases a.

These effects have been investigated by substituting the experimental values of f' and L′, obtained with the same velocity in equations (119) and (120).

In this way, from equation (119) the value of m is determined, where, from equations (117) and (135),

$$a_x = (a_0 + m\text{L}')ne^{\cdot 0222(\text{T}_x - \text{T}_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

And the equation (120) gives the effects of the load on the value of the constant A. After trial, however, it appears that the effects of the load upon the constant A are small so long as the loads are moderate, and that the diminution of the resistance with the increased load is explained by the value obtained for m from equation (119). From this equation, taking $\text{L}'_1\ f'^1$, $\text{L}'_2\ f'^2$, simultaneous values of L′ and f', and assuming T_x independent of the load

$$\frac{a_0 + m\text{L}'_1}{a_0 + m\text{L}'_2} = \frac{f'^2}{f'^1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

which gives the value of m.

The slight irregularities in the experiments affect the values of m thus found to a considerable extent, and a mean has been taken, which is

$$m = \cdot 002a_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (145)$$

Putting $\text{T}_x = 70{\cdot}5$, $\text{T}_0 = 60$, $a_x = \cdot 00074$, when L′=100, from equation (143)

$$a_0 + m\text{L}' = \cdot 0005861n.$$

Therefore

$$a_0 = \cdot 0004885n \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

The value for a thus obtained is therefore

$$a = \cdot 0004885n(1 + \cdot 002\,L')e^{\cdot 0222(T_x + T_m - T_0)} \quad . \quad . \quad . \quad (147)$$

and for the experiments with the bath at 90° Fahr.

$$a = \cdot 0004885n(1 + \cdot 002\,L')e^{\cdot 0222(T_m + 10\cdot 5)}$$

or

$$a = \cdot 0006161n(1 + \cdot 002\,L')e^{\cdot 0222T_m} \quad . \quad . \quad . \quad (148)$$

From equation (148) and the values of T_m, Table III., the values of

$$\frac{a}{n}$$

for Mr. TOWER'S experiments have been calculated.

Putting

$$\mu = \cdot 00001e^{-\cdot 0221T_m} \quad . \quad . \quad . \quad (149)$$

and, substituting in equations (47) from (148) and (149),

$$\left.\begin{aligned} -K_1 &= \frac{66\cdot 08\,Ne^{-\cdot 0665T_m}}{(1 + \cdot 002\,L')^2 n^2} \\ -K_2 &= \frac{\cdot 0034\,Ne^{-\cdot 0443T_m}}{(1 + \cdot 002\,L')n} \\ \frac{K_2}{K_1} &= \cdot 00005139(1 + \cdot 002\,L')e^{\cdot 0222T_m} \end{aligned}\right\} \quad . \quad . \quad . \quad (150)$$

for the circumstances of Mr. TOWER'S experiments, to which the equations of Sections VI. and VII. then become applicable.

40. *Application of the Equations to the Circumstances of* Mr. TOWER'S *Experiments on Brass No.* 1, *given in Table I., Art.* 34, *to determine* c, ϕ_0, ϕ_1, f' *and* $p - p_0$.

The circumstances are, *the unit of length being the inch,*

$$\left.\begin{aligned} &R = 2 \text{ (inches)} \\ &\theta = 78°\ 31'\ 20'' \\ &\frac{a_0}{n} = \cdot 0004885, \text{ already deduces, equation (146)} \\ &\left.\begin{aligned} L &= 484\,L' \\ U &= \cdot 2094 \end{aligned}\right\} \text{ equations (124)} \\ &T_0 = 60, \text{ assumed} \\ &T_x = 70\cdot 5, \text{ equation (140)} \\ &T_m = \text{tabular values, Table III., the increase with load being neglected} \end{aligned}\right\} \quad . \quad (151)$$

For a first Approximation as long as c is small.

Equations (94), (89), (91), and (95) are used to determine c, ϕ_0, ϕ, f'_1 for the experiments in Table I. (TOWER), these being made with brass No. 1.

Putting as in equation (124)

$$f' = \frac{nM}{2R^2}$$

equation (94) gives

$$f_1' = -1\cdot37\ K_2 n,$$

and by equation (150)

$$f_1' = \cdot004658 \frac{Ne^{-0443T_m}}{1 + 002\,L'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

From equation (152) the values of f' to a first approximation have been calculated, using the values of T_m given in Table III. These are given as f_1' in Table IV.

TABLE IV.—Olive Oil, Brass No. 1.

Length of the journal	6 inches.
Chord of the arc of contact of the brass	3·92 inches.
Radius of the journal	2 inches.
Temperature of the oil bath	90° Fahr.
" surrounding objects	60° Fahr. (assumed).
Difference in radii of brass and journal at 60°	0·0006 inch (deduced).
Effect of necking or variations in radius to increase friction	1·25.

L' the nominal load in lbs. per square inch, being the total load divided by 24.
N the number of revolutions per minute.
f' the nominal friction in lbs. per square inch from Table I. in Mr. TOWER's first Report (see Art. 34).
(f_1') the nominal friction calculated by complete approximation for $c=5$ (see Art. 40, equation (159)).
f_2' the nominal friction calculated to a second approximation, equation (154).
f_1' the nominal friction calculated to a first approximation, equation (152).
c the ratio of the distance between the centres of the brass and journal to the difference in the radii, equation (153).
a the difference in the radii of the brass and journal (see equation (157)).
ϕ_1 the angular distance from the middle of the arc of contact of the point of maximum pressure, equation (91).
$\phi_0 - \frac{\pi}{2}$ the angular distance from the middle of the arc of contact of the point of nearest approach (see equation (89)).
T_m the rise in temperature of the film of oil owing to the friction, equation (120).

N.		100	150	200	250	300	350	400	450
T_m Fahr.		3·45	5·83	8·13	10·02	11·77	13·26	14·46	15·37
$L'=415$	f'	..	·498	·580	·622	·705	·787	·870	·995
	(f_1')	..	..	(·57)	(·65)	..	..	..	f_2' ·108
	f_1'	..	·300	·360	·414	·460	·504	·544	·589
	c	..	·67	·578	·520	·487	·457	·436	·413
	a	·00154	·00162	·0017	·00178	·00182	·00187	·00191	·00194
	ϕ_1	..	..	−7° 0′ 0″	−7° 0′ 0″	..	..	..	..
	$\phi_1 - \frac{\pi}{2}$	..	..	−42° 0′ 0″	−42° 0′ 0″	..	..	..	..
$L'=363$	f'	..	·472	·580	·616	·689	·725	·798	·907
	(f_1')	.	(·498)	..	..	..	..	..	f_2' ·920
	f_1'	·233	·314	·378	·434	·482	·526	·573	·616
	c	..	·520	·462	·408	·380	·357	·340	·312
	a	..	·00151	·00161	·00168	·00171	·00177	·00181	·00183
	ϕ_1	..	−7° 0′ 0″	..	..	..	..	..	..
	$\phi_0 - \frac{\pi}{2}$	..	−42° 0′ 0″	..	..	..	..	..	..

TABLE IV.—Olive Oil, Brass No. 1—continued.

N.		100	150	200	250	300	350	400	450
T_m Fahr.		3·45	5·83	8·13	10·02	11·77	13·26	14·46	15·37
L′=310	f'	..	·464	·526	·588	·650	·684	·742	·835
	(f_1')	(·370)	..	..	..	..	f_2' ·765	·805	·845
	f_1'	·249	·336	·404	·464	·517	·582	·610	·660
	c	·510	·392	·337	·305	·284	·266	·254	·234
	a	·00138	·00144	·00151	·00158	·00161	·00166	·00172	·00172
	ϕ_1	− 7° 0′ 0″	..	..	..	..	..	..	..
	$\phi_0 - \frac{\pi}{2}$	−42° 0′ 0″	..	..	..	..	..	..	..
L′=253	f'	·361	·436	·514	·592	·644	·669	·747	·789
	f_2'	..	..	..	·62	·670	·712	·760	·810
	f_1'	·266	·358	·431	·495	·550	·600	·650	·707
	c	·377	·287	·242	·224	·200	·195	·180	·171
	a	·00128	·00135	·00141	·00148	·00151	·00156	·00159	·00161
	ϕ_1	..	..	..	..	..	..	..	..
	$\phi_0 - \frac{\pi}{2}$	..	..	..	..	..	..	..	..
L′=205	f'	·368	·430	·512	·572	·613	·675	·736	·818
	f_2'	·380	·457	·530	·595	·65	·701	·755	·810
	f_1'	·285	·385	·464	·534	·592	·646	·700	·755
	c	·255	·195	·165	·152	·141	·132	·126	·118
	a	·00119	·00126	·00131	·00138	·00240	·00145	·00148	·00135
	ϕ_1	..	..	..	..	..	..	−1° 13′ 0″	−1° 5′ 0″
	$\phi_0 - \frac{\pi}{2}$	..	..	..	..	..	..	−60° 0′ 0″	−62° 0′ 0″
L′=152	f'	·351	·458	·535	·611	·672	·718	·764	·871
	f_2'	·352	·458	·530	·601	·665	·717	·778	·840
	f_1'	·307	·414	·498	·574	·638	·695	·753	·813
	c	·165	·126	·107	·098	·089	·086	·082	·075
	a	·00111	·00116	·00122	·00128	·00130	·00134	·00137	·00139
	ϕ_1	..	−1° 13′ 0″	−1° 0′ 0″	−0° 57′ 0″	−0° 50′ 0″	−0° 44′ 0″	−0° 38′ 0″	−0° 31′ 0″
	$\phi_0 - \frac{\pi}{2}$	..	−61° 0′ 0″	−65° 0′ 0″	−27° 20′ 0″	−69° 0′ 0″	−69° 40′ 0″	−70° 20′ 0″	−72° 0′ 0″
L′=100	f'	·360	·450	·550	·630	·690	·770	·820	·890
	f_2'	·352	·465	·555	·637	·708	·770	·831	·897
	f_1'	·336	·463	·546	·628	·697	·760	·823	·890
	c	·090	·0691	·0600	·0541	·0492	·0471	·0460	·0420
	a	·00101	·00106	·00112	·00116	·00120	·00123	·00125	·00127
	ϕ_1	−0° 43′ 0″	−0° 26′ 0″	−0° 17′ 0″	−0° 10′ 0″	−0° 4′ 0″	−0° 2′ 30″	−0° 0′ 0″	+0° 5′ 0″
	$\phi_0 - \frac{\pi}{2}$	−68° 30′ 0″	−73° 20′ 0″	−75° 20′ 0″	−76° 30′ 0″	−77° 20′ 0″	−77° 40′ 0″	−0° 78′ 0″	−78° 40′ 0″

As compared with the experimental values f' given in the Table IV., it is seen that the agreement holds as long as c is less than ·06, after which, as c increases, the values of f_1' become too small, or while the values of f_1' continue to diminish as the load and a increase, the experimental values of f' after diminishing till c is about ·1 or ·15 begin to increase again. In order to see how far this law of variation was explained by the theory, it was necessary to find f_2' the values of f' to a second approximation, and before this to obtain the values of c.

Putting, as in equation (124),

$$L = 4{\cdot}84\, L',$$

equation (94) gives

$$c = -2{\cdot}059\frac{L'}{K_1};$$

and by equation (150)

$$c = {\cdot}03116(1+{\cdot}1002\,L')^2\frac{n^2L'}{N}e^{{\cdot}0665T_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

Equation (153) gives the values of

$$\frac{c}{n^2}.$$

To obtain the value of n from the experiments, these values of $\frac{c}{n^2}$ are substituted in the equation for f', retaining the squares of c, which obtained from equations (89) and (85), is

$$f_2' = f_1'(1+5c^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

whence, substituting the values of $\frac{c}{n^2}$ obtained from (153) we have

$$f_2' = f_1' + 5\left(\frac{c}{n^2}\right)^2 n^4 f_1'.$$

Therefore, choosing any experimental values of f', and subtracting the corresponding value of f_1' in Table IV., n is given by—

$$n^4 = \frac{f'-f_1'}{5\left(\frac{c}{n^2}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

In this experiment irregularities become important, and it has been necessary to calculate several values of n in this way and take the mean, which is

$$n = 1{\cdot}25 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

It has been shown (Art. 35) that necking might account for a value of n as great as 1·33, while if there were no necking n would still have a value in consequence of variations of a along the journal.

Substituting this value of n in equations (148 and 153),

$$\left.\begin{aligned} a &= {\cdot}00077(1+{\cdot}002\,L')e^{{\cdot}0222T_m} \\ c &= {\cdot}0487(1+{\cdot}002\,L')\frac{2L'}{N}e^{{\cdot}0665T_m} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

from which equation the values of a and c have been calculated for Table IV. for all values of L′ less than 415 lbs. These are all Mr. Tower's experiments with olive oil, except those of which Mr. Tower has expressed himself doubtful as to the results.

The values of c, as given by equation (94) are only a first approximation and are too large, but the error is not large, even when $c=\cdot 5$ only amounting to 8 per cent., as is shown by comparing equation 104 with 95.

With these values of c in the equation (154) the values of f_2' have been calculated for all values of c up to ·250. At $c=\cdot 12$ these values of f_2' are about 5 per cent. larger than the experimental values, but they have been carried to $c=\cdot 25$ in order to show that the calculated friction follows in its variations the idiosyncracies of the experimental frictions, falling with the load to a certain minimum, and then rising again.

These values of f_2' carry the comparison of the frictions deduced from the theory up to loads of 205 lbs. for all velocities, and up to 363 lbs. for the highest velocity. To carry the approximation further use has been made of the more complete integrations of the equations for the case of

$$c=\cdot 5.$$

These are given by equations (104) and (105).

As already stated, comparing (104) with (94) it appears that when $c=\cdot 5$ the approximate values of c in the Table IV. are about 8 per cent. too large; that is to say, a value $c=\cdot 540$ in the table would show that the actual value was $c=\cdot 5$.

Comparing equation (95), from which the values f_1' have been calculated, with equation (105), it appears that when $c=\cdot 5$ the values of f' by (105) are given by

$$f'=\frac{2\cdot 3773}{1\cdot 37}f_1'$$

This is not, however, quite satisfactory, as that portion of the friction which is due to necking does not increase with the load. This portion in f_1' is

$$\frac{n-1}{n}f_1'$$

So that for $c=\cdot 5$

$$f'=\frac{2\cdot 3773+1\cdot 37(n-1)}{1\cdot 37n}f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (158)$$

and since $n=1\cdot 25$, this gives for $c=\cdot 5$

$$f'=1\cdot 585f_1'$$

If, therefore, any of the approximate values of c were exactly ·540, the complete value of f' would be 1·585 times the value of f_1'. This does not happen, the nearest approximate values of c being ·578, ·520, ·520, ·510. Multiplying the corresponding values of f_1' by 1·585, the results are as follows :—

Tabular. c.	Tabular. f_1'.	$1·505\, f_1'$.	Experimental. f.	Difference.
·578	·36	·57	·58	·01
·520	·414	·656	·65	$-$·005
·520	·314	·498	·472	$-$·026
·510	·249	·394		

It thus appears that the approximation is very close, the calculated values for the first, in which c is greater than ·540, being too small, and for the rest, in which c was smaller than ·540, too large, which is exactly what was to be expected.

These corrected values of f_1' have been introduced in Table IV. in brackets. As they occur with different loads and different velocities, they afford a very severe test of the correctness of the conclusions arrived at as to the variations of A and T with the load and temperature, also as to the condition expressed by n. Had the values of c and f' been completely calculated as for the case of $c=·5$, there would have been close agreement for all the calculated and experimental values of f'.

This close agreement strongly implies, what was hardly to be expected, namely, that the surfaces, in altering their form under increasing loads, preserve their circular shape so exactly that the thickness of the oil film is everywhere approximately

$$a(1+c\sin(\theta-\phi)).$$

A still more severe test of this is, however, furnished by the pressure experiments with brass No. 2 in Mr. TOWER's second report.

41. *The Velocity of Maximum Carrying Power.*

The limits to the carrying powers are not very clearly brought out in these recorded experiments of Mr. TOWER, as indeed it was impossible they should be, as each time the limit is reached the brass and journal require refitting. But it appears from Table I. and all the similar tables with the oil bath in Mr. TOWER's reports, that the limit was not reached in any case in which the load and velocity were such as to make c less than ·5. In many cases they were such as to make c considerably greater than this, but in such cases there seems to have been occasional seizing. There seems, however, to have been one exception to this case, in which the journal was run at 20 revolutions per minute with a nominal load of 443 lbs. per square inch with brass No. 2 without seizing, in which case c, as determined either by the friction or load, becomes nearly ·9.

It does not appear that any case is mentioned of seizing having occurred at high speeds, so that the experiments show no evidence of a maximum carrying power at a particular velocity.

This is so far in accordance with the conclusions of Art. (33*a*), for substituting the values of A B C E, as determined Art. (37), it appears by equation (123B) that the maximum would not be reached until T_m, the rise of temperature due to friction, reached 72° Fahr., which, seeing that at a velocity of 450 revolutions T_m is less than 17°, implies that the maximum carrying power would not be reached until the speed was 1500 or 2000 revolutions; notwithstanding $\frac{dL}{dU}c$ constant is very small at 450 revolutions.

This is with the rise of temperature due to legitimate friction with perfect lubrication. But if, owing to inequalities of the surfaces, there is excessive friction without corresponding carrying power, *i.e.*, if n, the effect of necking, is as large as 3 or 4, which it is with new brasses, then the maximum carrying power might be reached at comparatively small velocities; thus suppose $T=13$ when $N=100$ $U=21$, Equation (123B) gives

$$\frac{dL}{dU}=0,$$

or the maximum carrying power would be reached; all which seems to be in strict accordance with experience, particularly with new brasses.

42. *Application of the Equations to* Mr. Tower's *Experiments with Brass No.* 2 *to determine the Oil Pressure round the Journal.*

The approximate equation (74) is available to determine the pressure at any part of the journal, *i.e.*, for any value of θ so long as c is small, but these approximations fail for much smaller values of c than for others, for this reason, together with the fact that the only case in which the pressure has been measured c is large, the pressures have only been calculated for $c=\cdot 5$, in which the approximations have been carried to the extreme extent.

These are obtained directly from equation (107), and the pressures divided by $K_1 c$ are given in Table II., Section VI.

The results of Mr. Tower's experiments with brass No. 2 are given in Table XII., Art. 34.

Had the friction been recorded in the experiments in which Mr. Tower measured the pressures with brass No. 2 as with brass No. 1, the values of c might have been obtained as in the case of brass No. 1. But as this was not done the value of c for these experiments with brass No. 2 could only be inferred from the agreement of the relative oil pressures measured in different parts of the journal, those calculated for the same parts with a particular value of c. This was a matter of trial, and as it was found that the agreement was very close when

$$c=\cdot 5,$$

further attempts were not made.

With the section at the middle of the brass the calculated and experimental results are shown in Table V.

TABLE V.—Comparison of Relative Pressures, calculated by Equation (107) when $c = \cdot 5$, with the Pressures Measured by Mr. TOWER, see Table XII., Art. 34, Brass No. 2.

The values of θ measured from middle of arc at which pressures were measured.	Pressure measured at the middle of the journal. Table XII., TOWER.	$\frac{p-p_0}{-K_1c}$ calculated. Table II.	Relative values, experimental.	Relative values, calculated.	$-K_1c$.
$-29^\circ\ 20'\ 20''$	500	·7923	·800	·781	639
0 0 0	625	1·0150	1·000	1·000	615
29 20 20	370	·6609	·592	·651	560

This agreement, although not exact, is, considering the nature of the test, very close. The divergence seems to show that in the experiments c was somewhat more than ·5, but it is doubtful if the agreement would have been exact as, owing to the journal having been run in one direction only, it seems probable that the radius of the brass was probably slightly greatest on the *on* side.

Deducing the value of K_1c by dividing the experimental pressure by the calculated values of $\frac{p-p_0}{K_1c}$ the values given in the last column are found. An alteration in the value of c would but slightly have altered the middle value of $\frac{p-p_0}{Kc}$ in the same direction as the alteration of c; hence taking this value, and making $c = \cdot 520$, as being nearer the real value,

$$K_1c = -640 \qquad (159)$$

In these experiments

$$N = 150$$

$$L' = 333 \qquad (160)$$

From equation (104)

$$L = -2\cdot5504 \times Kc$$

$$\frac{L}{4} = 408 \qquad (161)$$

therefore

$$s = \frac{L}{4L'}$$

$$= 1\cdot21 \qquad (162)$$

To find a

$$K_1 = -1230$$

by equation (150)

$$=\frac{66{\cdot}08e^{-{\cdot}0665T_m}}{n^2(1+002L)^2}N$$

whence

$$a_x^2=000001088e^{-{\cdot}0665\,T_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (163)$$

and taking T_m the same as with brass No. 1, and olive oil at N=180, *i.e.*, 5·83° Fahr., with brass No. 2, at 70·5° Fahr.

$$\left.\begin{array}{r} a_x=00086 \\ \text{instead of with brass No. 1} \qquad a_n=00077 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (164)$$

The difference in the radii of curvature of the two brasses, the one deduced from the measured friction, the other deduced from the measured differences of pressure at different positions round the journal, come out equal within $\frac{1}{10000}$th part of an inch, and the values of a differing only by 11 per cent. Had the frictions been given with brass No. 2, this agreement would have afforded an independent comparison of the values of a. As it is, the only probability of equality in these two brasses arises from the probability of their having been bedded in the same way.

In deducing the value a for brass No. 2, it has been assumed that the oil, which was mineral, had the same law of viscosity as the olive oil. Both these oils were used with brass No. 1, and the results are nearly the same, the mean resistances, as given by Mr. TOWER, are as 0·623 to 0·654, or the viscosity of the mineral oil being 0·95 that for olive oil; had this been taken into account, the value of a_x for brass No. 2 would have been still nearer that for brass No. 1, being ·00084 as against ·00077.

As the radii of the two brasses seem to be so near, and as the resistance was measured for brass No. 1 under circumstances closely resembling those of the experiment with No. 2, a further test of the exactness of the theory is furnished by comparing the calculated friction with brass No. 2 with that measured with brass No. 1, with the same oil, the same speed, and nearly the same load.

As in equation (158)

$$-f'=2{\cdot}3773K_2+1{\cdot}37(n-1)K_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (165)$$

$$-K_2={\cdot}21\frac{\mu}{a}N$$

Whence, taking account of the values of μ for mineral and olive oils, and the values of a for brass No. 1 and No. 2 for mineral oil and brass No. 2, K_2 has 0·871 of the value in equation (150)

$$-K_2=\frac{{\cdot}871\times{\cdot}0034e^{-{\cdot}0443T_m}}{(1+{\cdot}002L')n}N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (166)$$

which, when $T=5\cdot83°$, $N=150$, $L'=337$, $n=1\cdot25$, being substituted in equation

$$K_2=-0\cdot1665$$
$$f'=4\cdot46 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (167)$$

In Mr. Tower's Table IV., it appears that with brass No. 1, mineral oil,

$$N=150 \qquad L'=310 \qquad f'=4\cdot4 \qquad L'=415 \qquad f'=\cdot51$$

whence interpolating for

$$L'=337$$
$$f'=4\cdot58 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

This agreement is very close, for taking account of the difference of radius, the calculated friction for brass No. 2 should have been about ·95 of the measured friction with brass No. 2.

In order to show the agreement between the calculated pressures and those of Mr. Tower, the values of $\frac{p-p_0}{K_1c}$ for $c=\cdot5$ have been plotted, and are shown in Plate 8, figs. 2 and 3, the crosses indicating the experiments with brass No. 2, as in Table VII. (Tower).

43. *Conclusions.*

The experiments to which the theory has been definitely applied may be taken to include all Mr. Tower's experiments with the 4-inch journal and oil bath, in which the number of revolutions per minute was between 100 and 450, and the nominal loads in lbs. per sq. inch between 100 and 415. The other experiments with the oil bath were with loads from 415, till the journal seized at 520, 573, or 625 ; and a set of experiments with brass No. 2 at 20 revolutions per minute. All these experiments were under extreme conditions, for which, by the theory, c was so great as to render lubrication incomplete, and preclude the application of the theory without further integrations.

The theory has, therefore, been tested by experiments throughout the entire range of circumstances to which the particular integrations undertaken are applicable. And the results, which in many cases check one another, are consistent throughout.

The agreement of the experimental results with the particular equations obtained on the assumption that the brass, as well as the journal, are truly circular, must be attributed to the same causes as the great regularity presented by the experimental results themselves.

Fundamental amongst these causes is, as Mr. Tower has pointed out, the perfect supply of lubricant obtained with the oil bath. But scarcely less important must have been the truth with which the brasses were first fitted to the journal, the smallness of the subsequent wear, and the variety of the conditions as to magnitude of load, speed, and direction of motion.

That a brass in continuous use should preserve a circular section with a constant radius requires either that there should be no wear at all, or that the wear at any point P should be proportional to sin (90°—POH).

Experience shows that there is wear in ordinary practice, and even in Mr. Tower's experiments there seems to have been some wear. In these experiments, however, there is every reason to suppose that the wear would have been approximately proportional to $c \sin(\phi_0-\theta)=c \sin(90°-\mathrm{POH})$, because this represents the approach of the brass to the journal within the mean distance a, for all points, except those at which it is negative; at these there would be no wear. So long then as the journal ran in one direction only, the wear would tend to preserve the radius and true circular form of that portion of the arc from C to F (fig. 17, p. 193), altering the radius at F, and enlarging it from F to D. On reversal, however, C and F change sides, and hence alternate motion in both directions would preserve the radius constant all over the brass. The experience emphasized by Mr. Tower that the journal after running for some time in one direction would not run at first in the other, strongly bears out this conclusion. Hence it follows that had the journal been continuously run in one direction the condition of lubrication, as shown by the distribution of oil pressure round the journal, would have been modified, the pressure falling between O and B on the *on* side of the journal, a conclusion which is borne out by the fact that in the experiments with brass No. 2, which was run for some time continuously in one direction, the pressure measured on the *on* side is somewhat below that calculated on the assumption of circular form, although the agreement is close for the other four points (see fig. 2, Plate 8).

When the surfaces are completely separated by oil it is difficult to see what can cause wear. But there is generally metallic contact at starting, and hence abrasion, which will introduce metallic particles into the oil (blacken it); these particles will be more or less carried round and round, causing wear and increasing the number of particles and the viscosity of the oil. Thus the rate of wear would depend on the impurities in the oil, the values of c, $\frac{1}{a}$ and the velocity of the journal, and hence would render the greatest velocity at which the maximum load could be carried with a large value of c small. A conclusion which seems to be confirmed by Mr. Tower's experiments at twenty revolutions per minute.

In cases such as engine bearings the wear causes the radius of curvature of the brass continually to increase, and hence a and c must continually increase with wear. But in order to apply the theory to such cases the change in the direction of the load (or U_1 and V_1) have to be taken into account.

That the circumstances of Mr. Tower's experiments are not those of ordinary practice, and hence that the particular equations deduced in order to apply the theory to these experiments do not apply to ordinary cases, does not show that the general theory, as given in equations (15), (18), and (19), could not be applied to ordinary cases were the conditions sufficiently known.

These experiments of Mr. TOWER have afforded the means of verifying the theory for a particular case, and hence has established its truth as applicable to all cases for which the integrations can be effected.

The circumstances expressed by

$$\mu, \frac{\mathrm{L}}{\mathrm{U}}, \frac{a}{\mathrm{R}}, c, \phi_0, \phi_1, n, m, \mathrm{C}, \mathrm{A}, \mathrm{E}, \mathrm{B},$$

which are shown by the theory to be the principal circumstances on which lubrication depends, although not the same in other cases, will still be the principal circumstances and indicate the conditions to be fulfilled in order to secure good lubrication.

The verification of the equations for viscous fluids under such extreme circumstances affords a severe test of the truth and completeness of the assumptions on which these equations were founded. While the result of the whole research is to point to a conclusion (important in Natural Philosophy) that not only in cases of intentional lubrication, but wherever hard surfaces under pressure slide over each other without abrasion, they are separated by a film of some foreign matter whether perceivable or not. And that the question as to whether this action can be continuous or not turns on whether the action tends to preserve the matter between the surfaces at the points of pressure as in the apparently unique case of the revolving journal, or tends to sweep it to one side as is the result of all backwards and forward rubbing with continuous pressure.

The fact that a little grease will enable almost any surfaces to slide for a time has tended doubtless to obscure the action of the revolving journal to maintain the oil between the surfaces at the point of pressure. And yet, although only now understood, it is this action that has alone rendered our machines, and even our carriages possible. The only other self-acting system of lubrication is that of reciprocating joints with alternate pressure on and separation (drawing the oil back or a fresh supply) of the surfaces. This plays an important part in certain machines, as in the steam engine, and is as fundamental to animal mechanics as the lubricating action of the journal is to mechanical contrivances.

Fig. 3. Fig. 2.

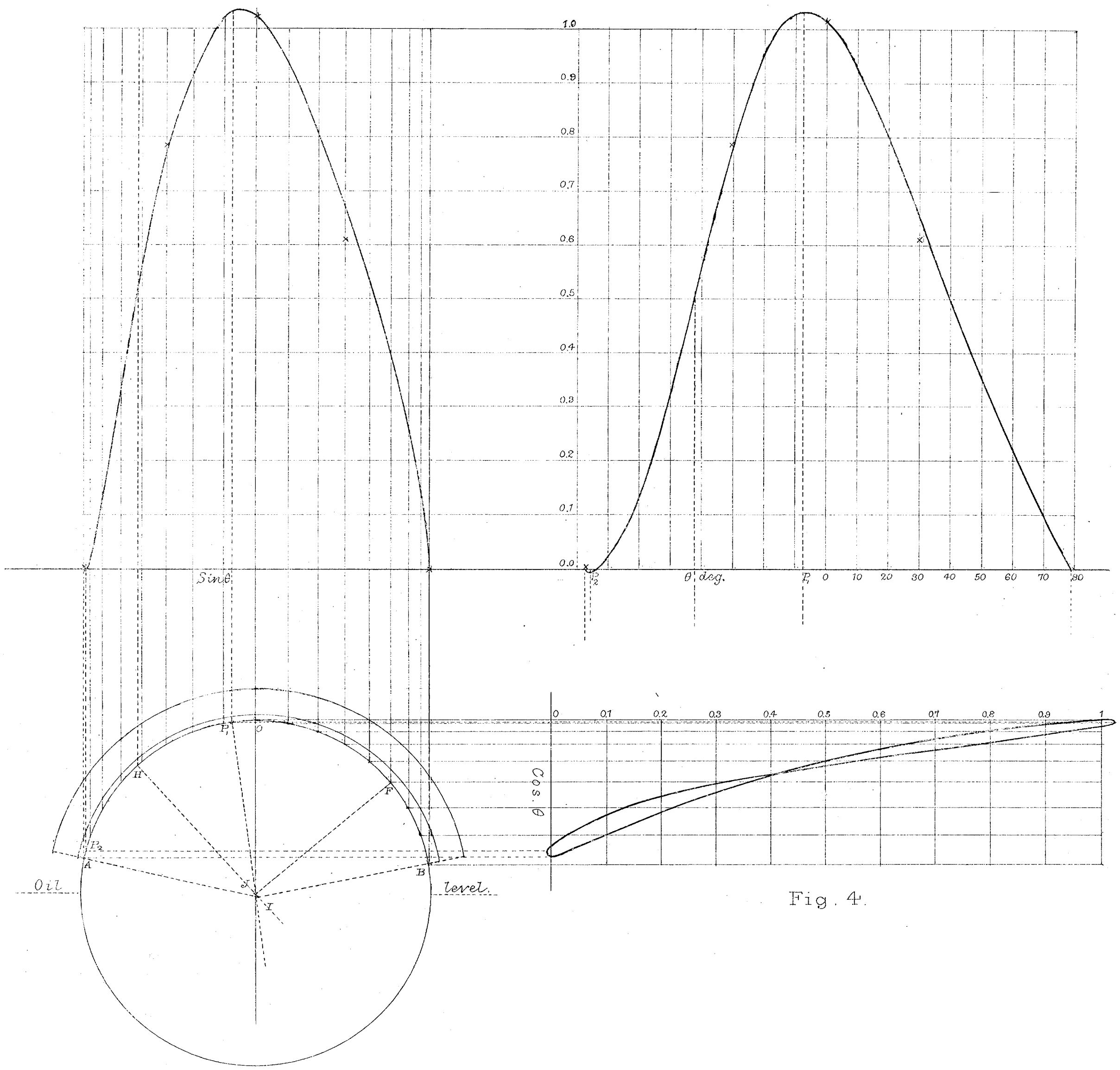


Fig. 4.

Fig. 1.

West, Newman & Co. sc.